IOWA STATE TEACHERS COLLEGE
FEB 9 1961
LIBRARY

AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

WEI-LIANG CHOW
THE JOHNS HOPKINS UNIVERSITY

J. A. DIEUDONNÉ
INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

A. M. GLEASON
HARVARD UNIVERSITY

PHILIP HARTMAN
THE JOHNS HOPKINS UNIVERSITY

WITH THE COÖPERATION OF

L. V. AHLFORS
A. BOREL
H. CARTAN
S. S. CHERN
C. CHEVALLEY
K. IWASAWA
K. KODAIRA
F. I. MAUTNER
J. MILNOR
A. WEIL

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume LXXXII, Number 4
OCTOBER, 1960

THE JOHNS HOPKINS PRESS
BALTIMORE 18, MARYLAND
U. S. A.

CONTENTS

PAGE

Solvable Lie groups acting on nilmanifolds. By LOUIS AUSLANDER, . . 653
The manifolds covered by Riemannian homogeneous manifold. By JOSEPH A. WOLF, ... 661
Fundamental groups of compact solvmanifolds. By LOUIS AUSLANDER, . 689
Compact transformation groups of S^n with an $(n-1)$-dimensional orbit. By HSIEN-CHUNG WANG, ... 698
Conformal transformations of Riemann surfaces. By LEON GREENBERG, . 749
Subgroups of the modular group and sums of squares. By MORRIS NEWMAN, 761
Abelian extensions of differential fields. By E. R. KOLCHIN, ... 779
The algebraic determination of the genus of knots. By LEE NEUWIRTH, . 791
An ultrahyperbolic equation with an integral condition. By O. G. OWEN, . 799
Quadratic forms over fields of characteristic 2. By CHIH-HAN SAH, . . 812
A generalized theorem of Krull-Seidenberg on parametrized algebras of finite type. By HEISUKE HIRONAKA, ... 831
The Wedderburn decomposition of commutative Banach algebras. By WILLIAM G. BADÉ and PHILIP C. CURTIS, JR., ... 851
Functions that are harmonic or zero. By JOHN W. GREEN, ... 867
Fundamental groups on a Lorentz manifold. By J. WOLFGANG SMITH, . 873
Analytic and algebraic dependence of meromorphic functions. By REINHOLD REMMERT, ... 891
Hereditarily compact spaces. By A. H. STONE, ... 900
Imbedding of holomorphically complete complex spaces. By RAGHAVAN NARASIMHAN, ... 917
Deformations of compact differentiable transformation groups. By RICHARD S. PALAIS and THOMAS E. STEWART, ... 935

The AMERICAN JOURNAL OF MATHEMATICS appears four times yearly.

The subscription price of the JOURNAL is $11.00 in the U. S.; $11.30 in Canada and $11.60 in other foreign countries. The price of single numbers is $3.00.

Manuscripts intended for publication in the JOURNAL should be sent to Professor W. L. CHOW, The Johns Hopkins University, Baltimore 18, Md.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE 18, MARYLAND, U. S. A.

THE JOHNS HOPKINS PRESS supplies to the authors 100 free reprints of every article appearing in the AMERICAN JOURNAL OF MATHEMATICS. On the other hand, neither THE JOHNS HOPKINS PRESS nor the AMERICAN JOURNAL OF MATHEMATICS can accept orders for additional reprints. Authors interested in securing more than 100 reprints are advised to make arrangements directly with the printers, J. H. FURST CO., 109 MARKET PLACE, BALTIMORE 2, MARYLAND.

The typescripts submitted can be in English, French, German or Italian and should be prepared in accordance with the instructions listed on the inside back cover of this issue.

Second-class postage paid at Baltimore, Maryland.

PRINTED IN THE UNITED STATES OF AMERICA
BY J. H. FURST COMPANY, BALTIMORE, MARYLAND

SOLVABLE LIE GROUPS ACTING ON NILMANIFOLDS.*

By Louis Auslander.[1]

Introduction. In [1] Maurice Auslander and the author classified all the connected, simply connected solvable Lie groups which can operate transitively and with discrete isotropy group on the n-dimensional torus. If Z^n denotes the additive group of integers taken n times, the above classification is equivalent to the classification of all the connected, simply connected solvable Lie groups S which can contain Z^n as discrete uniform subgroup. If R^s denotes the direct sum of the additive group of reals taken s times, our result can be stated as follows: A necessary and sufficient condition for a connected, simply connected solvable Lie group S to contain Z^n as a discrete uniform subgroup is that S satisfy the diagram

$$1 \to R^s \to S \to R^t \to 1,$$

where $s + t = n$, the automorphisms of R^s induced by R^t form a compact group and the extension is the *split* extension. If T denotes the torus group of automorphisms of the group R^sZ^n, then T is uniquely extendable to a torus group of automorphisms of R^n and S is easily seen to be a subgroup of $R^n \cdot T$, where the dot denotes the semi-direct product.

This paper has as its goal the extension of the above results to the case where Z^n is replaced by the fundamental group of any compact nilmanifold. We will always use Γ to denote such a group. Or restating our goals, we will study the connected, simply connected, solvable Lie groups S which can contain the group Γ as a discrete uniform subgroup. By a result of Malcev [3] there exists a *unique* connected, simply connected nilpotent group $N(\Gamma)$ which contains Γ as discrete uniform subgroup. We will first show that for a connected, simply connected solvable Lie group S containing Γ as discrete uniform subgroup there exists a torus group T of automorphisms of $L(\Gamma)$ such that $S \subset N(\Gamma) \cdot T$, where the dot again denotes the semi-direct product. This will be the easier part of our program. The second part will be devoted

---

* Revised October 1, 1959.

[1] During part of the time this paper was being written the author was supported by a National Science Foundation Grant.

to showing that the group extension is essentially controlled by the discrete uniform subgroup. The explicit form of our result is given in Theorem 2.

Once we have proven these results for the fundamental groups of nilmanifolds it is reasonable to try and extend these results to the fundamental groups of compact solvmanifolds. We did not succeed in this attempt. This may perhaps best be explained by remarking that our whole proof of Theorem 2 rests on a series of steps which enable us to reduce a non-abelian group extension problem to the abelian group extension problem which had already been solved in [1]. Hence, the work in [1], rather than being replaced by this paper, is made to form the basis of the results here obtained. However, for the general case of solvmanifolds, our reduction fails.

For the rest of this paper we will use the expression solvable Lie group to denote connected, simply connected solvable Lie groups.

1. Theorem 1.

THEOREM 1. *Let S be a solvable Lie group and contain Γ as discrete uniform subgroup. Further, let Γ be a discrete uniform subgroup of the nilpotent group N. Then there exists a torus group T of automorphisms of N such that $S \subset N \cdot T$, where the dot denotes the semi-direct product.*

Proof.[2] Let H be the maximal normal analytic nilpotent subgroup of S. Then ΓH is normal in S and by the results in [3, 4] can be identified with a normal subgroup of N, for it contains the commutator subgroup of both N and S.

The inner automorphisms of S induce automorphisms of ΓH, and hence isomorphisms of Γ into N, and hence by [3] isomorphisms of N onto N. Thus S operates on N. We now form the semidirect product $N \cdot S$ and let Δ denote the subgroup of elements (x, x^{-1}) for $x \in \Gamma H$. Then Δ is normal in $N \cdot S$ and N and S have isomorphic images in $N \cdot S/\Delta$ under the natural projection. Further, the image of N is a normal subgroup and the factor group of $N \cdot S/\Delta$ by the image of N is isomorphic to the toroid group $S/\Gamma H$. Since any extension of a connected, simply connected solvable group by a compact group is a split extension, we have $N \cdot S/\Delta \approx N \cdot T$, with T a torus group. Hence S has an isomorphic image in $N \cdot T$.

2. Statement of Theorem 2. Let G and H be connected Lie groups. An extension of G by H is a pair (E, π) consisting of a connected Lie group

[2] The proof of this theorem was suggested to the author by the referee and is a considerable improvement of the author's original proof.

E which contains G as a closed normal subgroup and a continuous homomorphism of E onto H whose kernel is G. Let $A(G)$ denote all continuous automorphisms of G and let $I(G)$ denote the normal subgroup of $A(G)$ consisting of inner automorphisms. Then if (E,π) is an extension of G by H, this determines a homomorphism ϕ of H into $A(G)/I(G)$. We will call this homomorphism the *action* of H on G. We will also say that the group extension (E,π) is associated with the triple (G,H,ϕ) and vice versa. Clearly there may be many or no group extensions associated with a given triple.

Let Γ be a non-abelian uniform subgroup of a connected, simply connected, nilpotent Lie group N. Let S be any connected, simply connected, solvable Lie group which contains Γ as a discrete uniform subgroup. Let H be the maximal normal analytic nilpotent subgroup of S. Then H contains the commutator subgroup of S and is a closed normal subgroup of S. Let π: $S \to S/H$ and let $V = S/H$. Then V is a connected, simply connected, abelian Lie group and V contains $\Gamma H/H$ as a discrete uniform subgroup. Hence associated with the group extension (S,π) is the triple (H,V,ϕ), where ϕ is the action of V on H.

THEOREM 2. *Let S_1 and S_2 be connected, simply connected, solvable Lie groups with discrete uniform subgroups Γ_1 and Γ_2 and let Γ_1 and Γ_2 be fundamental groups of compact nilmanifolds. Assume that α is an isomorphism of Γ_1 onto Γ_2 with the following properties:*

1. *If H_1 and H_2 are the maximal normal analytic nilpotent subgroups of S_1 and S_2 respectively, then α induces an isomorphism α_1 of $\Gamma_1 \cap H_1$ onto $\Gamma_2 \cap H_2$. By the results in* [3], *we see that α is uniquely extendable to an isomorphism $\alpha^*{}_1$ of H_1 onto H_2.*

2. *Let α induce the isomorphism α_2 of $\Gamma_1/\Gamma_1 \cap H_1$ onto $\Gamma_2/\Gamma_2 \cap H_2$. Then α_2 can be extended to give a unique isomorphism $\alpha^*{}_2$ of V_1 onto V_2. Further, if S_1 and S_2 are associated with the triples (H_1,V_1,ϕ_1) and (H_2,V_2,ϕ_2) respectively, we shall assume that the following diagram is commutative.*

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi_1} & A(H_1)/I(H_1) \\ \Big\downarrow{\scriptstyle \alpha^*{}_2} & & \Big\downarrow{\scriptstyle \alpha^{**}{}_1} \\ V_2 & \xrightarrow{\phi_2} & A(H_2)/I(H_2), \end{array}$$

where $\alpha^*{}_1$ is defined in the obvious way from $\alpha^*{}_1$.

Under all the above assumptions S_1 is isomorphic to S_2.

3. Preliminaries to Theorem C. Let S contain Γ as a discrete uniform subgroup, where Γ is a discrete uniform subgroup of the nilpotent Lie group N. Let H be the maximal normal analytic nilpotent subgroup of S and let $H^* = H\Gamma$. Then H^* is a nilpotent group and we will denote by H^*_j the j entry in the lower central series of H^*. Let $C(H^*_j)$ denote the closure of H^*_j. Let r be the largest index such that $C(H^*_r)$ has a non-trivial identity component. Let H_r denote the identity component of $C(H^*_r)$.

LEMMA 1. *H_r is in the center of H^* and H_r is a normal analytic subgroup of S.*

Proof. Let $h \in H^*$ and let D be the subset of H^* consisting of all elements $hxh^{-1}x^{-1}$ for $x \in H_r$. Clearly D is a connected set and contains the identity element. We assert that D is a subset of $C(H^*_{r+1})$. To see this, notice that the mapping $\phi(x) = hxh^{-1}x^{-1}$ is a continuous mapping of H^* into itself. Hence for any subset A of H^*

$$\phi(C(A)) \subset C(\phi(A)),$$

where $C(\ )$ denotes the closure of the set in the parenthesis. Hence

$$\phi(C(H^*_r)) \subset C(\phi(H^*_r)) \subset C(H^*_{r+1}).$$

But $C(H^*_r) \supset H_r$. This proves the assertion that $D \subset C(H^*_{r+1})$. Hence, since the component of the identity is trival in $C(H^*_{r+1})$ and D is connected and contains the identity, we see that D must consist of the identity element. Hence H is in the center of H^*.

It remains to show that H_r is a normal subgroup of S. Now H^* is a normal subgroup of S and H^*_r is a characteristic subgroup of H^*, i.e., every automorphism of H^* leaves H^*_r invariant. Hence every continuous automorphism of H^* leaves $C(H^*_r)$ invariant. Hence $C(H^*_r)$ is normal in S. From this it follows that the identity component of $C(H^*_r)$, H_r, is normal in S.

LEMMA 2. *$H_r \cap \Gamma$ is a discrete uniform subgroup in H_r.*

Proof. $\Gamma_0 = H \cap \Gamma$ is a discrete uniform subgroup of H. But Γ_0 is a subgroup of N also. This induces an isomorphism of H into N. We will call this image of H in N also. Consider $C(H^*_r) \subset N$. Then $H^*_r \subset N_r$, where N_r is the r entry in the lower central series of N, and, since N_r is closed, $C(H^*_r) \subset N_r$ also. Let Γ_r be the r entry in the lower central series in Γ. Then $\Gamma_r \subset H^*_r$ and hence in $C(H^*_r)$. Now Γ_r is normal in N_r (see [3]) and hence in $C(H^*_r)$. Since $C(H^*_r)$ is locally connected, H_r is open and

closed in $C(H^*_r)$. Thus $\Gamma_r H_r$ is closed in $C(H^*_r)$ and hence in N_r. It follows that Γ_r is uniform in $\Gamma_r H_r$, that is $\Gamma_r H_r/\Gamma_r$ is compact. Hence $\Gamma_r \cap H_r$ is uniform in H_r.

COROLLARY 1. *S/H_r has a discrete uniform subgroup isomorphic to $\Gamma H_r/H_r$. Further, H/H_r has a discrete uniform subgroup isomorphic to $\Gamma/H_r\Gamma$.*

The proof of this corollary and Lemma 3 below follow easily from Lemma 2.

LEMMA 3. *Let S_1, S_2 and Γ_1, Γ_2 satisfy the hypothesis of Theorem 2. Then*

$$S_1/(H_1)_r, \qquad S_2/(H_2)_r$$

and

$$\Gamma_1/(H_1)_r \cap \Gamma_1 \qquad \Gamma_2/(H_2)_r \cap \Gamma_2$$

satisfy the hypothesis to Theorem 2.

LEMMA 4. *Let S contain Γ as a discrete uniform subgroup. Then S contains the commutator subgroup of N as a normal subgroup.*

Proof. The statement of the lemma is meaningful as we have identified H in S with H in N. Further, H contains the commutator of N and hence by the identification of H in S with H in N, this gives rise to a group in S. The proof of this lemma will be induction on the dimension of S. Clearly the lemma is true if dim. $S = 1$. Let us assume now that the lemma is true for dim. $S < n$. Assume now that dim. $S = n$. Then consider $H_r \subset S$ and the exact sequence

$$1 \to H_r \xrightarrow{i} S \xrightarrow{j} S/H_r \to 1.$$

Then by the previous discussion we see that S/H_r satisfies the induction hypothesis and hence the commutator subgroup C^* of N/N_r is a normal subgroup of S/H_r. Hence $j^{-1}(C^*) = C$ is a normal subgroup of S. But considering $H \subset S$, we get the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & H_r & \longrightarrow & H & \longrightarrow & H/H_r \longrightarrow 1 \\ & & \downarrow & & \downarrow k & & \downarrow \\ 1 & \longrightarrow & H_r & \longrightarrow & N & \xrightarrow{j^*} & N/H_r \longrightarrow 1, \end{array}$$

where $k^*(C^*)$ is the commutator subgroup of N/H_r and $j^{*-1}(k^*(C^*)) = k(C)$. Since H_r is contained in the commutator subgroup of N, $k(C)$ is the commutator subgroup of N. This proves the lemma.

4. Theorem 2.

Proof of Theorem 2. It is clear from the hypothesis that dim. S_1 = dim. S_2. We will prove the Theorem by induction on the dimension of S_1. The theorem is trivially true if dim. $S_1 = 1$. Assume the theorem is true if dim. $S_1 < n$, and assume that the dim. $S_1 = n$. In order to simplify the presentation and notation, we will use the isomorphisms stated in the hypothesis to this theorem to identify objects; i. e. we will use H to denote H_1 and H_2, H_r to denote $(H_1)_r$, $(H_2)_r$ and Γ to denote Γ_1 and Γ_2, etc. Whenever this is done we will explicitly be using the isomorphisms in the hypothesis to this theorem. We may then form the groups S_1/H_r and S_2/H_r. As a consequence of Lemma 3 and our induction hypothesis we have that S_1/H_r and S_2/H_r are isomorphic. We will identify them under this isomorphism and denote them by S^*. Hence we have

$$1 \to H_r \to S_i \to S^* \to 1, \qquad i = 1, 2,$$

where H_r is abelian.

LEMMA 5. *We can choose cocycles f_1 and f_2 representing the group extensions S_1 and S_2 respectively such that:*

a) $f_1 - f_2 = 0$ *when restricted to* $\Gamma H/H_r \subset S^*$

b) *Corresponding to* $f_1 - f_2 = f$ *there is a* Lie group *satisfying the diagram:*

$$1 \to H_r \to S' \to S^* \to 1$$

with S^1 *having* f *as extension cocycle.*

Proof. In order to keep the exposition at a minimum we will assume in presenting the details of this proof that the reader is familiar with the ideas in Hochschild's paper [2]. In particular, we will prove the lemma by constructing two continuous cocyles or factor sets f_1 and f_2 such that $f_1 - f_2 = 0$ when restricted to $\Gamma H/H_r$.

Consider $H\Gamma \subset S_i$ and $H\Gamma/H_r \subset S^*$. Now a factor set may be generated by a cross section in the bundle S_i over the base space S^* with fiber H_r. Further, if the cross sections are continuous, then the factor sets corresponding to them will be continuous. Now for S_i, $i = 1, 2$, choose continuous sections $\rho^*_1 = \rho^*_2$ over $H\Gamma/H_r \subset S^*$ with images in $H\Gamma$. Then since the fiber H_r is homeomorphic to euclidean space, each of the two partial sections can be extended to continuous sections ρ_1 and ρ_2 respectively. Let f_i be the factor set associated with the section ρ_i. Then clearly f_i is continuous and $f_1 - f_2 = 0$

when restricted to $H\Gamma/H_r$ since $\rho_1 = \rho_2$ when restricted to $H\Gamma/H_r$. This proves the lemma.

Since H_r is in the center of ΓH, $H/H_r = S^*$ acts trivially on H. Further, f restricted to $\Gamma H/H_r$ is cohomologous to zero. From these considerations, we see that S' has a discrete uniform subgroup Γ' such that Γ' is also a discrete uniform subgroup of the nilpotent Lie group $N' = H_r \oplus N/H_r$, where $\oplus$ denotes the direct product of groups.

Let G be the pre-image of H/H_r in S'. Then H_r is a normal subgroup in G.

LEMMA 6. *$G = H_r \oplus K$, where K is also a normal subgroup in S' and S'/K has a free abelian discrete uniform subgroup.*

(We will now complete the proof of Theorem C under the assumption that Lemma 6 is true. We will present a proof of Lemma 6 at the end of the demonstration of Theorem 2.)

By Lemma 6, we have that K is a normal subgroup of S' and K is a pre-image of H/H_r in S'. Hence we may form

$$1 \to H_r \to S'/K \to V \to 1,$$

where S'/K has a discrete abelian uniform subgroup. Therefore, by the results in [1], S'/K is a split extension.

Now consider the diagram

$$\begin{array}{ccccccc} 1 \to & H_r & \to & S' & \to & S^* & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & H_r & \to & S'/K & \to & S^*/K & \to 1. \end{array}$$

This shows that S' is also a split extension and hence $f_1 - f_2$ is cohomologous to zero. Hence S_1 is isomorphic to S_2.

Proof of Lemma 6. By Lemma 4, the commutator subgroup C of N' is a normal subgroup of S' and $C \cap \Gamma'$ is a discrete uniform subgroup of C. Now $C \subset G$ and intersects H_r in the identity element. Hence we may form

$$1 \to G/C \to S'/C \to V \to 1.$$

Now G/C is abelian and the image of Γ' in S'/C is a discrete uniform subgroup. Further, the image of Γ' in V is a discrete uniform subgroup which acts trivially on G/C. If $i(H_r)$ denotes the image of H in G/C, there is a normal subgroup K^* in G/C intersecting $i(H_r)$ in the identity and invariant

under V acting on G/C. This follows from the fact that V acts as a compact group on G/C and $i(H_r)$ is an invariant space of V acting on G/C. Hence K^* is normal in S'/C. Let K be the pre-image in S' of K^* in S'/C. Clearly K satisfies the conclusions of the Lemma provided S'/C has a discrete abelian uniform subgroup, it is a split extension, i.e. there exists V^*, a subgroup of S'/C, which is mapped isomorphically onto V by the projection mapping. Further let Γ^* be any pre-image of Γ' in V^*. Then Γ^* acts trivially on $i(H_r)$ in G/C.

Choose a discrete uniform subgroup of the image of $i(H_r)$ in S'/K, and consider the image of Γ^* in S'/K. Then these groups generate a discrete uniform abelian subgroup in S'/K.

REFERENCES.

[1] L. Auslander and M. Auslander, "Solvable Lie groups and locally Euclidean Riemann spaces," to appear in the *Proceedings of the American Mathematical Society*.

[2] G. Hochschild, "Group extensions of Lie groups," *Annals of Mathematics*, vol. 54 (1951), pp. 96-109.

[3] A. Malcev, "On a class of homogeneous spaces," *American Mathematical Society Translations*, No. 39.

[4] G. D. Mostow, "Factor spaces of solvable groups," *Annals of Mathematics*, vol. 60 (1954), pp. 1-27.

THE MANIFOLDS COVERED BY A RIEMANNIAN HOMOGENEOUS MANIFOLD.*

By JOSEPH A. WOLF.[1]

Introduction. The sphere is known to be the universal covering for complete connected Riemannian manifolds of constant positive curvature. More precisely, if M is an n-dimensional complete connected Riemannian manifold of constant sectional curvature $k^2 > 0$ with $k > 0$, and if S^n is the sphere of radius k^{-1} in Euclidean space R^{n+1}, with the induced metric, then there is a covering of M by S^n such that the covering projection is a local isometry. Because of this phenonmenon, the complete connected Riemannian manifolds of constant positive curvature are called the "spherical space-forms." In his thesis, G. Vincent [14] attempted to classify them. Following this line of investigation, we take a compact connected Riemannian homogeneous manifold M and ask which Riemannian manifolds admit M as a Riemannian covering manifold. In Chapter I, this problem is reduced to a problem on discrete subgroups of compact Lie groups:

Given a compact Lie group G and a closed subgroup K, find all finite subgroups Γ of G such that Γ meets the union of the conjugates of K only at the identity element of G.

For the most part we restrict our attention to the case where Γ lies in the identity component of G, or, equivalently, where G is connected. In Chapter II we obtain some bounds on the ranks of abelian subgroups of Γ, and see that the problem of classifying these groups Γ is inaccessible unless rank. G $-$ rank. $K \leqq 1$.

Chapter II ends with a sharper bound on the ranks of abelian subgroups of Γ, in case rank. G $-$ rank. $K = 1$, which implies that every abelian subgroup of Γ is cyclic if the semisimple part of G is simply connected and Γ lies in the identity component of G. We remark that H. Zassenhaus [16] and M. Suzuki [13] have given a complete classification of the finite groups

* Received August 25, 1959; revised May 23, 1960.

[1] This work is part of the author's doctoral dissertation, and was supported by National Science Foundation predoctoral fellowships.

with all abelian subgroups cyclic. Under certain conditions on G, K and the order of Γ (Corollary 10.1) it follows that Γ itself is cyclic.

In Chapter III we obtain an arithmetic criterion (Theorem 6), assuming G and K connected, for an arbitrary given finite cyclic subgroup of G to act freely on G/K. This criterion involves the Weyl group of G and the position of K in G. It is applied to an arbitrary finite subgroup Σ of G by finding cyclic subgroups of Σ such that every element of Σ is $\mathrm{ad}(G)$-conjugate to an element of one of these cyclic subgroups. Applying to the case where G is a classical group we obtain some information on elements of order 2 in Γ, assuming $\mathrm{rank}.\, G - \mathrm{rank}.\, K = 1$. Finally, we sharpen the bound on the ranks of abelian subgroups of Γ in case G is a special orthogonal group. In Chapter V we apply the arithmetic criterion again to the case where G is a classical group. Chapter VI is a consideration of the case where $\mathrm{rank}.\, G = \mathrm{rank}.\, K$, i.e., the Euler characteristic $\chi(M) \neq 0$, and shows that M covers only a finite number, up to isometry, of Riemannian manifolds.

Our problem can be considered as a generalization of the classical Clifford-Klein problem of finding all spherical space-forms, in that we have replaced the sphere by an arbitrary (for Theorems 1 to 4 and Theorem 6), or at least more general, compact connected Riemannian homogeneous manifold. Another direction of generalization is that of considering finite groups which admit a free topological action on a space similar in some way to a sphere. In this regard, we mention some of the work of P. A. Smith [11], P. E. Conner [5], J. Milnor [8] and A. Heller [6].

I especially wish to thank Professor S. S. Chern, under whose guidance this paper was written, for many helpful discussion and comments. I also wish to thank Professors A. Borel, H. C. Wang and R. S. Palais for many helpful discussions. Some of Professor Borel's work [1, 2] is crucial to this paper, and Professor Palais pointed out a lemma of Mostow used in the proof of Theorem 11.

Chapter I. Reduction to a problem on Lie groups.

I. 1. Covering spaces. In order to establish notation and terminology we will recall some well-known facts and definitions concerning covering spaces. All spaces will be Hausdorff and all maps will be continuous.

A *covering* is a map $p: X \to X'$ of arcwise connected, locally simply connected spaces where every element of X' has a neighborhood U such that p maps each component of $p^{-1}(U)$ homeomorphically onto U. $p^{-1}(x')$ is the *fibre* over $x' \in X'$. All fibres have the same cardinality, the *multiplicity* of

the covering. A *finite covering* is a covering of finite multiplicity. p induces a monomorphism of fundamental groups, and the covering is *normal* if $p \cdot \pi_1(X)$ is a normal subgroup of $\pi_1(X')$. This is independent of choice of basepoints. If H is a subgroup of $\pi_1(X')$, there is a covering $q: Y \to X'$ and a choice of basepoint in Y such that $q \cdot \pi_1(Y) = H$.

THEOREM 1. *If $p: X \to X'$ is a finite covering, there is a finite normal covering $q: X'' \to X$, where $pq: X'' \to X'$ is a finite normal covering.*

Proof. The multiplicity of p being equal to the index of $H = p \cdot \pi_1(X)$ in $\pi_1(X')$, the normalizer N of H in $\pi_1(X')$ has finite index in $\pi_1(X')$. Consequently ([7], p. 82) H has only a finite number of conjugates in $\pi_1(X')$, so ([7], p. 62) the intersection J of the conjugates of H has finite index in $\pi_1(X')$. Let $q: X'' \to X$ be a covering, where $q \cdot \pi_1(X'') = p^{-1}(J)$, and the normality conditions follows from the construction of J. *QED.*

An action of a group Γ on a space X is *effective* if the identity element of Γ is the only element inducing the identity transformation of X, is *free* if the identity element of Γ is the only element which leaves fixed a point of X, and is *properly discontinuous* if every point of X has a neighborhood which does not meet any of its transforms under Γ. If X is compact, the action of Γ is properly discontinuous if and only if Γ is finite and acts freely. The set $\Gamma(x)$, the *orbit* of a point x of X, is the set of images of x under Γ. The space X/Γ of orbits is given the quotient topology for the natural projection $x \to \Gamma(x)$; the natural projection $X \to X/\Gamma$ is a covering if and only if Γ acts properly discontinuously on X.

A *deck transformation* of a covering $p: X \to X'$ is a homeomorphism $\gamma: X \to X$, where $p \cdot \gamma = p$. The group Γ of all deck transformations acts properly discontinuously on X, and ([12], § 14) p is a normal covering if and only if Γ is simply transitive on each fibre, i.e., if and only if $p: X \to X'$ is a principal bundle with group Γ, i.e., if and only if $X' = X/\Gamma$.

A *Riemannian covering* is a covering $p: M \to M'$, where M and M' are Riemannian manifolds and p is a local isometry. If just one of M and M' is a Riemannian manifold, the requirement that p be a local isometry gives a Riemannian structure to the other and makes p a Riemannian covering. We easily see that a deck transformation is an isometry of M, because p is a local isometry.

A *Riemannian homogeneous manifold* is a Riemannian manifold whose group of isometries is transitive.

THEOREM 2. *If $q: M'' \to M$ is a Riemannian covering and M is Riemannian homogeneous, then M'' is Riemannian homogeneous.*

Proof. A one-parameter group of isometries of M is a homotopy and can be lifted to M'' by the covering homotopy theorem. The lifted homotopy consists of isometries of M'' because q is a local isometry. It follows that the group of isometries of M'' is locally transitive and therefore transitive. *QED.*

I. 2. Reduction to a problem on discrete subgroups of compact Lie groups.

THEOREM 3. *Let M be a compact connected Riemannian homogeneous manifold, G the group of isometries of M, K an isotropy subgroup of G, and Γ a subgroup of G. Then Γ is a properly discontinuous group of isometries of M if and only if Γ is finite and $\Gamma \cap \mathrm{ad}(G)K = 1$, where 1 is the identity element of G and $\mathrm{ad}(G)K$ is the set of all $\mathrm{ad}(g)k = gkg^{-1}$ with $g \in G$ and $k \in K$.*

Proof. M is compact and Γ is a group of isometries of M, so Γ is a properly discontinuous group of isometries of M if and only if Γ is finite and acts freely on M. G is transitive on M, so the isotropy subgroups of G are the subgroups $\mathrm{ad}(g)K$ with $g \in G$. Γ acts freely on M if and only if it meets each isotropy subgroup only at 1. Hence Γ acts freely if and only if $\Gamma \cap \mathrm{ad}(G)K = 1$. *QED.*

Using Theorems 1, 2 and 3, we see that the original problem

Given a compact connected Riemannian homogeneous manifold M, find all Riemannian manifolds which admit a Riemannian covering by M.

is reduced to the problem

Given a compact Lie group G and a closed subgroup K, find all finite subgroups Γ of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$.

by taking G to be the group of isometries of a finite Riemannian covering manifold of M and K to be an isotropy subgroup of G. We then note that G and K are both compact, each has only a finite number of components, and K meets every component of G.

Chapter II. Necessary conditions for a finite group to act as a properly discontinuous group of isometries of a compact connected Riemannian homogeneous manifold.

Given a compact Lie group G and a closed subgroup K, we will find some necessary conditions on finite subgroups Γ of G for $\Gamma \cap \mathrm{ad}(G)K = 1$. These will involve the ranks of G, K, and some subgroups of Γ.

The *rank* of a finite abelian group B is the minimal number of factors in a direct product decomposition of B into cyclic groups, and is denoted rank. B. For example, if p is a prime, the elementary p-group with p^h elements, $(Z_p)^h = Z_p \times \cdots \times Z_p$, the direct product of h copies of the cyclic group of order p, has rank h. The rank of B is the maximum of the ranks of its elementary p-subgroups, and B is cyclic if and only if rank. $B \leqq 1$. If p is a prime, the *p-rank* of B, denoted p-rank. B, is the rank of a p-Sylow subgroup of B. It is the maximal integer h such that B has a subgroup isomorphic to $(Z_p)^h$.

The *rank* of a compact Lie group H, denoted rank. H, is, as usual, the common dimension of the maximal toral subgroups of H.

II. 1. A bound on the ranks of certain abelian subgroups.

THEOREM 4. *Let K be a closed subgroup of a compact Lie group G.*

1. *Given a finite subgroup Γ of G such that $\Gamma \cap \mathrm{ad}(G) = 1$ and an abelian subgroup B of Γ which lies in a torus of G, we have* rank. $B \leqq$ rank. G $-$ rank. K.

2. *The above bound is the best possible in the sense that there is a positive integer $m(G,K)$ such that, given a finite abelian group A with* rank. $A \leqq$ rank. $G -$ rank. K *and $m(G,K)$ prime to the order of A, a torus of G has a subgroup A' which is isomorphic to A and such that $A' \cap \mathrm{ad}(G)K = 1$.*

Proof. Let T' be a maximal torus of K, T a maximal torus of G which contains T', $n =$ rank. G and $k =$ rank. K. We replace B by a conjugate which lies in T and still have $B \cap \mathrm{ad}(G)K = 1$, hence $B \cap T' = 1$. It follows that the canonical map of T onto the $(n-k)$-torus T/T' maps B monomorphically. Since a finite subgroup of an $(n-k)$-torus has rank at most $n-k$, we conclude rank. $B \leqq n-k$. This proves the first statement.

Let K_0 be the identity component of K, G_0 the identity component of G, $W = \{w_1, \cdots, w_q\}$ an enumeration of the Weyl group of G_0 with respect

to T, and $\{a_1, \cdots, a_t\}$ a set of automorphisms of G_0 which preserve T such that the automorphism group $\mathrm{ad}(G)$ of G_0 can be written as the union of the $a_i \cdot \mathrm{ad}(G_0)$. Two elements of T are $\mathrm{ad}(G_0)$-conjugate if and only if they are W-conjugate, and it follows that an element of T lies in $\mathrm{ad}(G)K_0$ if and only if it lies in one of the $T_{ij} = a_i(w_j(T'))$. As there are only a finite number of the k-tori T_{ij}, there exists an $(n-k)$-torus V in T which intersects each T_{ij} in a finite group. Let $m(G,K)$ be the product of the primes occurring in the orders of these finite groups and in the order b of K/K_0. Let $\beta \in V$ have order prime to $m(G,K)$ and lie in $\mathrm{ad}(G)K$. Then $\beta^b \in \mathrm{Ad}(G)K_0$, so $\beta^b \in T_{ij}$ for some (i,j). Since the order of β^b is also prime to $m(G,K)$, this implies, by the definition of $m(G,K)$, that $\beta^b = 1$. Since the order of β is prime to b, this implies $\beta = 1$.

We can find a subgroup A' of V which is isomorphic to A because V is an $(n-k)$-torus and A is a finite abelian group of rank at most $n-k$. The considerations above show that $A' \cap \mathrm{ad}(G)K = 1$ if the order of A, hence of A', is prime to $m(G,K)$. *QED.*

In Chapter III we will see examples where K, and even G, is connected and $m(G,K)$ must be even, hence $m(G,K) > 1$.

II. 2. The work of A. Borel on torsion and subgroups which lie in a torus. A. Borel has proved ([1], Chapter XII) that if G is a compact connected Lie group with classifying space B_G, p is a prime, and the integral cohomology ring $\boldsymbol{H}^*(B_G, \boldsymbol{Z})$ has no p-torsion, then every elementary p-subgroup (subgroup isomorphic to some $(Z_p)^h$) of G lies in a torus of G. A case by case check proves the converse. Borel has also shown that $\boldsymbol{H}^*(G, \boldsymbol{Z})$ has p-torsion if and only if $\boldsymbol{H}^*(B_G, \boldsymbol{Z})$ has p-torsion, using known results and checking the case $p = 5$ for $\boldsymbol{E}_6$, $p = 5$ and $p = 7$ for $\boldsymbol{E}_7$ and $p = 7$ for $\boldsymbol{E}_8$. The summary of the situation is that the following are equivalent:

1. $\boldsymbol{H}^*(G, \boldsymbol{Z})$ has no p-torsion.
2. $\boldsymbol{H}^*(B_G, \boldsymbol{Z})$ has no p-torsion.
3. G_{ss} being the semisimple part of G, $\boldsymbol{H}^*(G_{ss}, \boldsymbol{Z})$ has no p-torsion.
4. $\pi_1(G_{ss})$ has order prime to p, and, if G' is a simple factor of the universal covering group of G_{ss}, then $\boldsymbol{H}^*(G', \boldsymbol{Z})$ has no p-torsion.

Finally, if H is a compact, connected, simple, simply connected Lie group, then $\boldsymbol{H}^*(H, \boldsymbol{Z})$ has p-torsion in precisely these cases:

1. $p = 2$ and $H = \boldsymbol{E}_8$, $\boldsymbol{E}_7$, $\boldsymbol{E}_6$, $\boldsymbol{F}_4$, $\boldsymbol{G}_2$, or $\boldsymbol{Spin}(n)$ with $n \geqq 7$.

2. $p=3$ and $H=\boldsymbol{E}_8$, $\boldsymbol{E}_7$, $\boldsymbol{E}_6$, or $\boldsymbol{F}_4$.

3. $p=5$ and $H=\boldsymbol{E}_8$.

An immediate consequence of Theorem 4 and this work of A. Borel is:

THEOREM 4'. *Let G be a compact Lie group of rank n, K a closed subgroup of* rank k, *and Γ a finite subgroup of the identity component G_0 of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$. Then if p is a prime for which $\boldsymbol{H}^*(G_0, \boldsymbol{Z})$ has no p-torsion, every abelian subgroup of Γ has p-*rank $\leqq n-k$. *If $\boldsymbol{H}^*(G_0, \boldsymbol{Z})$ is torsion-free, every abelian subgroup of Γ has* rank $\leqq n-k$.

It is now clear that the problems in applying Theorem 4 are closely related to the existence of p-torsion in G. This is of two sorts—p-torsion from the fundamental group of G and p-torsion from the simply connected versions of the simple factors of G. Finally, we can only hope to classify our groups Γ in case rank. G — rank. $K \leqq 1$, due to the present rate of the theory of finite groups. We will see, however, that p-torsion in G is of little importance in case rank. G = rank. K, and that only the p-torsion from $\pi_1(G)$ is of importance in case rank. G — rank. $K = 1$.

In addition to the results mentioned above, A. Borel has shown [2]

Let G be a compact connected Lie group, $\pi_1(G)$ torsion-free and $x \in G$. The centralizer of x in G is connected.

As the identity component of the centralizer of x in G is the union of the maximal tori of G which contain x, it easily follows, if $\pi_1(G)$ is torsion-free, that every abelian subgroup of G with 2 generators lies in a torus of G. We will depend heavily on this result of A. Borel in the next section.

II. 3. A further bound on the ranks of abelian subgroups. The main purpose of this section is to prove:

THEOREM 5. *Let G be a compact connected Lie group, K a closed subgroup with* rank. G — rank. $K = 1$, *Γ a finite subgroup of G with $\Gamma \cap \mathrm{ad}(G)K = 1$, p a prime, and $h(p)$ the p-*rank *of $\pi_1(G)$. Then every abelian subgroup of Γ has p-*rank $\leqq h(p)+1$. *If $h(p)=2$, then every abelian subgroup of Γ has p-*rank $\leqq 2$.

We will first need a few lemmas. The first two of these lemmas are known, but not well-known, so it seems best to write them out.

LEMMA 5.1. *Let G be a compact connected Lie group, G_{ss} the semi-*

simple part of G, and $Z(G)_0$ the identity component of the center of G. There is a covering $\phi: G_{ss} \times Z(G)_0 \to G$ given by $\phi(g,t) = gt$. ϕ is an epimorphism of compact connected Lie groups and the kernel, ker. ϕ, *of ϕ is the set of all (g, g^{-1}) with $g \in G_{ss} \cap Z(G)_0$.*

Proof. G_{ss} has finite center and $G = G_{ss} \cdot Z(G)_0$. Note that ker. ϕ is finite and lies in the center of $G_{ss} \times Z(G)_0$.

LEMMA 5.2. *Let G be a compact connected Lie group. As a topological space, G is homeomorphic with $G_{ss} \times Z(G)_0$. As $Z(G)_0$ is a torus, it follows that the torsion subgroup of $\pi_1(G)$ is isomorphic to $\pi_1(G_{ss})$, and in particular p-rank. $\pi_1(G) = p$-rank. $\pi_1(G_{ss})$ for every prime p.*

Proof. We proceed by induction on the dimension s of the torus $Z(G)_0$, and the lemma is trivial if $s = 0$. If $s = 1$, we consider the principal bundle $G \to G/G_{ss} = Z(G)_0/(G_{ss} \cap Z(G)_0)$ with connected group G_{ss} and base which is a 1-sphere. Since this is a trivial bundle ([12], p. 99), G is homeomorphic to $G_{ss} \times$ (1-sphere), which is homeomorphic to $G_{ss} \times Z(G)_0$. Now assume $s > 1$. Take a subgroup H of G which is generated by G_{ss} and an $(s-1)$-torus T in $Z(G)_0$. H is homeomorphic to $G_{ss} \times T$ by induction. As before, the principal fibre bundle $G \to G/H$ tells us that G is homeomorphic to $H \times$ (1-sphere), so G is homeomorphic to $G_{ss} \times Z(G)_0$.

Now note that $\pi_1(G) = \pi_1(G_{ss}) \times \pi_1$(torus) and π_1(torus) is a free abelian group. *QED.*

LEMMA 5.3. *Let G be a compact connected Lie group, p a prime, and $h(p)$ the p-rank of $\pi_1(G)$. Let $\phi: G_{ss} \times Z(G)_0 \to G$ be the covering given by $\phi(g,t) = gt$, $\mu: G' \to G_{ss}$ the universal covering of G_{ss}, and $\theta: G' \times Z(G)_0 \to G$ the composition $\phi \cdot (\mu \times 1)$. Then every $(Z_p)^{h(p)+2}$ in G contains a $(Z_p)^2$ which is the θ-image of an abelian subgroup of $G' \times Z(G)_0$. If $h(p) = 2$, every $(Z_p)^3$ in G contains a $(Z_p)^2$ which is the θ-image of an abelian subgroup of $G' \times Z(G)_0$.*

Proof. Let $\phi\beta_1, \cdots, \phi\beta_{h(p)+2}$ generate a $(Z_p)^{h(p)+2}$ in G, $N = \ker. \phi$, and $\beta_j = (b_j, t_j) \in G_{ss} \times Z(G)_0$. As $[\beta_i, \beta_j] \in N$, where $[\ ,\ ]$ is the ordinary commutator, we know from the form of the elements of N that $[b_i, b_j] = 1 \in G_{ss}$. As $\beta_j^p \in N$ we also know that b_j^p is central in G_{ss}. Now take elements $c_j \in G'$ with $\mu(c_j) = b_j$. As b_j^p is central in G_{ss}, c_j^p is central in G'. Since the b_j commute with each other, the commutators $[c_i, c_j]$ lies in ker. μ and are thus central in G'.

Let u and v be elements of a group H such that $w = [u,v]$ commutes

with u. $uv = wvu$ and we assume $u^{n-1}v = w^{n-1}vu^{n-1}$ by induction on n. Hence $u^nv = uu^{n-1}v = uw^{n-1}vu^{n-1} = w^{n-1}uvu^{n-1} = w^{n-1}wvuu^{n-1} = wv^nu^n$ in general. In other words, $[u^n, v] = [u, v]^n$ if u commutes with $[u, v]$. Since $[c_i, c_j]$ is central in G', it commutes with c_i, and consequently $[c_i, c_j]^p = [c_i^p, c_j]$ which equals 1 because c_i^p is central in G'.

$M = \ker. \mu$ is isomorphic to $\pi_1(G_{ss})$ so, by Lemma 5.2 and the definition of $h(p)$, M does not contain a $(Z_p)^{h(p)+1}$. Now set $y_j = [c_{h(p)+2}, c_j]$ for $1 \leqq j \leqq h(p)+1$. We have just seen that $y_j^p = 1$. As $\mu[c_i, c_j] = [\mu c_i, \mu c_j] = [b_i, b_j] = 1$, $y_j \in M$. It follows that the y_j generate an elementary p-subgroup Y in M of rank at most $h(p)$. Since there are $h(p)+1$ of the y_j, we have a relation $y_1^{v_1}y_2^{v_2}\cdots y_{h(p)+1}^{v_{h(p)+1}} = 1$, v_j integers not all divisible by p. Set $c = c_1^{v_1}c_2^{v_2}\cdots c_{h(p)+1}^{v_{h(p)+1}}$ and $t = t_1^{v_1}t_2^{v_2}\cdots t_{h(p)+1}^{v_{h(p)+1}}$ and notice that the fact that $[c_i, c_jc_k] = [c_i, c_j]\cdot[c_i, c_k]$, a consequence of the fact that each $[c_i, c_q]$ is central in G', gives us $[c_{h(p)+2}, c] = 1$. We now have elements $\sigma = (c, t)$ and $\tau = (c_{h(p)+2}, t_{h(p)+2})$ in $G' \times Z(G)_0$ such that σ and τ generate an abelian group in $G' \times Z(G)_0$ whose θ-image is a $(Z_p)^2$ inside our original $(Z_p)^{h(p)+2}$ in G.

Now suppose $h(p) = 2$. As before, we have a $(Z_p)^3$ in G generated by $\phi\beta_1, \phi\beta_2, \phi\beta_3$; we have $\beta_j = (b_j, t_j)$; and we have $\mu(c_j) = b_j$. We set $y_1 = [c_1, c_2]$, $y_2 = [c_2, c_3]$, $y_3 = [c_3, c_1]$ and the y_j generate an elementary p-subgroup of M which, by definition of $h(p)$, has rank $\leqq 2$. This gives us a relation of the form $y_1^{v_1}y_2^{v_2}y_3^{v_3} = 1$, where the v_j are integers not all divisible by p. We can assume that v_1 is not divisible by p, so there are integers r and s such that $y_1 = y_2^r y_3^s$. If p divides s, $[c_1c_3^r, c_2] = 1$. If p doesn't divide s, there is an integer u with $us \equiv -r \pmod p$, and $[c_1c_2^u, c_2c_3^s] = 1$. In either case we get an abelian group from the c_j whose θ-image is a $(Z_p)^2$ inside our original $(Z_p)^3$ in G. *QED.*

Proof of Theorem 5. Let B be an abelian subgroup of Γ with p-rank. $B > h(p)+1$. Then B contains a $(Z_p)^{h(p)+2}$. By Lemma 5.3 we have a $(Z_p)^2$ in B which is the θ-image of an abelian subgroup S of $G' \times Z(G)_0$. S is generated by two elements. By a theorem of A. Borel, mentioned in § II.2, S lies in a torus of $G' \times Z(G)_0$, so $\theta(S)$ lies in a torus of G. Hence Γ contains a $(Z_p)^2$ which lies in a torus of G. As rank. G — rank. $K = 1$ and $\Gamma \cap \mathrm{ad}(G)K = 1$, this contradicts Theorem 4. The proof that $h(p) = 2$ implies p-rank. $B \leqq 2$ is identical. *QED.*

COROLLARY 5.1. *Let G be a compact connected Lie group which has torsion-free fundamental group, i.e., such that G_{ss} is simply connected. Let K be a closed subgroup of G such that* rank. G — rank. $K = 1$ *and let Γ be a*

finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$. Then every abelian subgroup of Γ is cyclic. The odd Sylow subgroups of Γ are cyclic and the 2-Sylow subgroups are cyclic or generalized quaternionic, i.e., given by two generators A and B with the relations

$$A^{2^{a-1}} = 1, \quad A^{2^{a-2}} = B^2, \quad BAB^{-1} = A^{-1}, \quad a \text{ integer}, \quad a > 2.$$

Proof. Let V be an abelian subgroup of Γ and write V as a product of p-subgroups. By Theorem 5, each of these p-subgroups has rank $\leqq 1$, hence is cyclic. Since V is a product of cyclic subgroups of pairwise relatively prime orders, it follows that V is cyclic. The rest is known ([14], Chapter I) to follow. *QED.*

Chapter III. An arithmetic criterion and first application to the classical groups.

III. 1. Angular parameters and the arithmetic criterion. Let G be a compact connected Lie group of rank n, K a closed connected subgroup of rank k, T a maximal torus of G which contains a maximal torus T' of K, $W = \{w_1, \cdots, w_q\}$ an enumeration of the Weyl group of G relative to T, and $T_i = w_i(T')$. We choose an *integral basis* of the Lie algebra T of T, i.e., an ordered basis $\mathsf{X} = \{X_1, \cdots, X_n\}$ of T such that $\exp(\sum_s a_s X_s) = 1$ if and only if each a_s is an integer. The Lie algebra T_i of T_i can be described as the set of all elements $\sum_s a_s X_s$ of T such that $\sum_s v_{ijs} a_s = 0$ for $1 \leqq j \leqq n-k$, where each $\{v_{ij1}, \cdots, v_{ijn}\} = V_{ij}$ is an ordered set of relatively prime integers. The v_{ijs} can be chosen to be rational because each T_i is closed in T, and the obvious normalization transforms each V_{ij} into a set of relatively prime integers.

DEFINITION. *The $q(n-k)$ ordered sets V_{ij} of relatively prime integers are the angular parameters of K in G relative to X.*

We remark that, for a given K and G, the choice of X does not specify the angular parameters of K in G uniquely.

Let Γ be a finite subgroup of G. We choose cyclic subgroups $\{\gamma_t\} = \Gamma_t$ of Γ such that every element of Γ is $\mathrm{ad}(G)$-conjugate to an element of one of the Γ_t. Then $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $\Gamma_t \cap \mathrm{ad}(G)K = 1$ for each t. The angular parameters of K in G give us an arithmetic criterion for $\Gamma_t \cap \mathrm{ad}(G)K = 1$:

THEOREM 6. *Let G be a compact connected Lie group of* rank n, *K a*

closed connected subgroup of rank k, $V_{ij} = \{v_{ij1}, v_{ij2}, \cdots, v_{ijn}\}$ *the angular parameters of* K *in* G *relative to an integral basis* $\mathsf{X} = \{X_1, \cdots, X_n\}$ *of the Lie algebra* T *of a maximal torus of* G, $B = \{\beta\}$ *a cyclic subgroup of order* m *in* G, *and* $X = \sum_s a_s X_s \in \mathsf{T}$ *such that* $\exp(X)$ *is* $\mathrm{ad}(G)$*-conjugate to* β. *Then each* $b_s = ma_s$ *is an integer, and* $B \cap \mathrm{ad}(G)K = 1$ *if and only if each*

$$V_i = \{m, \textstyle\sum_s v_{i1s} b_s, \sum_s v_{i2s} b_s, \cdots, \sum_s v_{i\,n-k\,s} b_s\}$$

is a set of relatively prime integers.

Proof. Each $b_s = ma_s$ is an integer because X is an integral basis of T and $\exp(mX) = \exp(\sum_s b_s X_s)$ is conjugate to $\beta^m = 1$.

We will use the notation leading to the definition of the angular parameters of K in G relative to X. An element of T lies in $\mathrm{ad}(G)K$ if and only if it lies in one of the T_i, so $B \cap \mathrm{ad}(G)K = 1$ if and only if $\exp(rX) \notin T_i$ for any i whenever $r \not\equiv 0 \pmod m$. X being an integral basis of T, $\exp(rX) \in T_i$ if and only if there is a choice a_{is} of integers such that $rX + \sum_s a_{is} X_s = \sum_s (ra_s + a_{is}) X_s$ lies in T_i, i.e., such that $\sum_s v_{ijs}(rb_s + ma_{is}) = 0$ for $1 \leqq j \leqq n-k$. Reducing modulo m this says that $r \sum_s v_{ijs} b_s \equiv 0$ for $1 \leqq j \leqq n-k$. If $r \not\equiv 0 \pmod m$, this implies that V_i is not a set of relatively prime integers.

Now suppose that V_i is not a set of relatively prime integers. Then there is an integer $r \not\equiv 0 \pmod m$ such that $r \sum_s v_{ijs} b_s \equiv 0 \pmod m$ for $1 \leqq j \leqq n-k$. We will show that $\exp(rX) \in T_i$, so $\beta^r \in \mathrm{ad}(G)K$. Let U_{ij} be the $(n-1)$-torus whose Lie algebra U_{ij} is the hyperplane $\sum_s v_{ijs} x_s = 0$ in T, where the x_s are coordinates relative to the basis X. $T_i = \bigcap_j U_{ij}$. V_{ij} being a set of relatively prime integers, we have integers c_{ijs} with $\sum_s c_{ijs} v_{ijs} = 1$. The congruences $r \sum_s v_{ijs} b_s \equiv 0 \pmod m$ gives us integers t_{ij} with $mt_{ij} + r \sum_s v_{ijs} b_s = 0$, so we have integers $a_{ijs} = c_{ijs} t_{ij}$ such that $\sum_s v_{ijs}(rb_s + ma_{ijs}) = 0$, for $1 \leqq j \leqq n-k$. This just says that $\exp(rX) \in U_{ij}$ for $1 \leqq j \leqq n-k$, so $\exp(rX) \in T_i$. *QED.*

Theorem 6 can be used to check $\Gamma \cap \mathrm{ad}(G)K = 1$ provided that the intersection of K with the identity component G_0 of G is connected and $\Gamma \subset G_0$. Let $\{f_t\}$ be automorphisms of G such that the automorphism group $\mathrm{ad}(G)$ of G is the union of the $\mathrm{ad}(G_0) \cdot f_t$. Let $K_t = f_t(K) \cap G_0$. Given $\Gamma \subset G_0$, $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $\Gamma \cap \mathrm{ad}(G_0)K_t = 1$ for every t. We can check the $\Gamma \cap \mathrm{ad}(G_0)K_t = 1$ with Theorem 6 and thus check $\Gamma \cap \mathrm{ad}(G)K = 1$.

The application of Theorem 6 is simplified when the Weyl group W of G acts on the integral basis X by signed permutations: the angular parameters can then be chosen so that each V_{ij} is obtained from V_{1j} by the same signed permutations. We will use this trick when G is a classical group.

III. 2. Even and odd subgroups of the classical groups. By the *classical groups* we mean the unitary groups $\boldsymbol{U}(n)$, the special unitary groups $\boldsymbol{SU}(n)$, the symplectic groups (often called the unitary symplectic groups) $\boldsymbol{Sp}(n)$, the special orthogonal groups $\boldsymbol{SO}(n)$, and the spin groups (universal covering groups of the special orthogonal groups) $\boldsymbol{Spin}(n)$. They are all compact connected Lie groups. $\boldsymbol{U}(n)$ has rank n, semisimple part $\boldsymbol{SU}(n)$ and fundamental group infinite cyclic. $\boldsymbol{SU}(n+1)$ has rank n, is simple for $n \geqq 1$, and is simply connected. $\boldsymbol{Sp}(n)$ has rank n, is simple for $n \geqq 1$, and is simply connected. $\boldsymbol{Sp}(n)$ can be viewed as all elements of $\boldsymbol{U}(2n)$ which preserve an antisymmetric nondegenerate 2-form on complex Euclidean space C^{2n}. Given an orthonormal basis $\{e_1, \cdots, e_{2n}\}$ of C^{2n}, we'll use the form

$$A\left(\sum_1^{2n} x_j e_j, \sum_1^{2n} y_j e_j\right) = \sum_1^n (x_j y_{j+n} - y_j x_{j+n}).$$

$\boldsymbol{SO}(2n \text{ or } 2n+1)$ has rank n. $\boldsymbol{SO}(k)$ is semisimple for $k \geqq 3$, simple for $4 \neq k \geqq 3$, has fundamental group Z_2 if $k \geqq 3$, and has universal covering group $\boldsymbol{Spin}(k)$.

Given a classical group G, we have a canonical choice of a maximal torus T of G:

1. $G = \boldsymbol{U}(n)$. T is the set of all matrices $\operatorname{diag}\{a_1, \cdots, a_n\}$, where each a_j is a unimodular complex number.

2. $G = \boldsymbol{SU}(n)$. T is the set of all $\operatorname{diag}\{a_1, \cdots, a_n\}$ of determinant 1, where each a_j is a unimodular complex number.

3. $G = \boldsymbol{Sp}(n)$. T is the set of all matrices $\begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix}$, where $\bar{D}$ is the complex conjugate of D and D is in the canonical maximal torus of $\boldsymbol{U}(n)$.

4. $G = \boldsymbol{SO}(2n \text{ or } 2n+1)$. T is the set of all matrices $\operatorname{diag}\{R(t_1), \cdots, R(t_n), (1)\}$, where $R(t) = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$ and the (1) appears only if $G = \boldsymbol{SO}(2n+1)$.

5. $G = \boldsymbol{Spin}(2n \text{ or } 2n+1)$. T is the complete inverse image of our chosen maximal torus in $\boldsymbol{SO}(2n \text{ or } 2n+1)$.

If G is not a ***Spin*** or special unitary group, we have a canonical choice of integral basis $\mathsf{X}_G = \{X_1, \cdots, X_n\}$ of the Lie algebra T of T:

1. $G = \boldsymbol{U}(n)$. $\exp(tX_j) = \operatorname{diag}\{1, \cdots, 1, \exp(2\pi i t), 1, \cdots, 1\}$, where the $\exp(2\pi i t)$ is in the j-th place and $i^2 = -1$.

2. $G = \boldsymbol{Sp}(n)$. $\exp(tX_j) = \begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix}$, where $\bar{D}$ is the complex conjugate of D and $D = \operatorname{diag}\{1, \cdots, 1, \exp(2\pi it), 1, \cdots, 1\}$ has the $\exp(2\pi it)$ in the j-th place.

3. $G = \boldsymbol{SO}(2n \text{ or } 2n+1)$. Let I_2 be the 2×2 identity matrix; $\exp(tX_j) = \operatorname{diag}\{I_2, \cdots, I_2, R(t), I_2, \cdots, I_2, (1)\}$, where $R(t)$ is the j-th block.

The Weyl group W of G acts on X_G by signed permutations:

1. $G = \boldsymbol{U}(n)$. W acts on X_G by all permutations.

2. $G = \boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n+1)$. W acts on X_G by all signed permutations.

3. $G = \boldsymbol{SO}(2n)$. W acts on X_G by all signed permutations where the the number of sign changes is even.

Let K be a closed connected subgroup of rank k in a classical group $G = \boldsymbol{U}(n)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n \text{ or } 2n+1)$. Replacing K by a conjugate, we have a maximal torus T' of K which lies in our canonical maximal torus T of G. The Lie algebra T' of T' is the intersection of $n-k$ hyperplanes $\sum_s v_{js} x_s = 0$, where the x_s are coordinates in T relative to the canonical integral basis X_G; we can assume that each $V_j = \{v_{j1}, v_{j2}, \cdots, v_{jn}\}$ is a set of relatively prime integers. If $W = \{w_1, \cdots, w_q\}$ is an enumeration of the Weyl group of G relative to T, W envisaged as a group of signed permutations on n-tuples from its action on X_G, the angular parameters of K in G relative to X_G are the $V_{ij} = w_i(V_j)$.

DEFINITION. *The $n-k$ ordered sets V_j of relatively prime integers are the canonical parameters of K in $G = \boldsymbol{U}(n)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n \text{ or } 2n+1)$.*

Let K be a closed connected subgroup of rank $k-1$ in $\boldsymbol{SU}(n)$. Viewing $\boldsymbol{SU}(n)$ as a subgroup of $\boldsymbol{U}(n)$, K has canonical parameters $V_1, \cdots, V_{n-k+1}$ in $\boldsymbol{U}(n)$. We may assume that $V_{n-k+1} = \{1, 1, \cdots, 1\}$.

DEFINITION. *The $n-k$ ordered sets $V_1, \cdots, V_{n-k}$ of relatively prime integers are the canonical parameters of K in $\boldsymbol{SU}(n)$.*

Let K be a closed connected subgroup of rank k in $\boldsymbol{Spin}(2n \text{ or } 2n+1)$ and let $f\colon \boldsymbol{Spin} \to \boldsymbol{SO}$ be the natural projection. We will use the canonical parameters $V_1, \cdots, V_{n-k}$ of $f(K)$ in $\boldsymbol{SO}(2n \text{ or } 2n+1)$ for the canonical parameters of K in $\boldsymbol{Spin}(2n \text{ or } 2n+1)$:

DEFINITION. *The $n-k$ ordered sets V_j of relatively prime integers are the canonical parameters of K in $\boldsymbol{Spin}(2n \text{ or } 2n+1)$.*

Given a closed connected subgroup K of a classical group G and an integral basis X of the Lie algebra of a maximal torus of G, we can always construct the angular parameters of K in G relative to X from the canonical parameters of K in G.

The fact that the Weyl group acts on the canonical parameters by signed permutations allows us to define:

DEFINITION. *Let K be a closed connected subgroup of a classical group $G = \boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, $\boldsymbol{SO}(2n$ or $2n+1)$ or $\boldsymbol{Spin}(2n$ or $2n+1)$ such that* rank. G — rank. $K = 1$. *Let $V = \{v_1, \cdots, v_n\}$ be the canonical parameter of K in G and set $v = v_1 v_2 \cdots v_n$. Then K is an even subgroup of G if v is an even integer; K is an odd subgroup of G if v is an odd integer.*

The most familiar examples of even subgroups are

$$\boldsymbol{U}(n-1) \subset \boldsymbol{U}(n),\ \boldsymbol{SU}(n-1) \subset \boldsymbol{SU}(n),\ \boldsymbol{Sp}(n-1) \subset \boldsymbol{Sp}(n),$$
$$\boldsymbol{SO}(2n-1) \subset \boldsymbol{SO}(2n) \text{ and } \boldsymbol{Spin}(2n-1) \subset \boldsymbol{Spin}(2n).$$

In these examples the canonical parameter can be taken to be $\{1, 0, \cdots, 0\}$.

III. 3. The orthogonal groups. If G is a classical group $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n+1)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{Spin}(2n$ or $2n+1)$ of rank n, K is a closed connected subgroup of rank $n-1$ and Γ is a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$, then Corollary 5.1 tells us that every abelian subgroup of Γ is cyclic. If, however, $G = \boldsymbol{SO}(2n$ or $2n+1)$, then we only know that every abelian subgroup of Γ is of the form $Z_u \times Z_v$, where u is a power of 2. As it is known [11] that a $(Z_2)^2$ cannot act freely on the sphere $S^{2n-1} = \boldsymbol{SO}(2n)/\boldsymbol{SO}(2n-1)$, there is, at least for some choices of K, room for improvement:

THEOREM 7. *Let G be a special orthogonal group $\boldsymbol{SO}(q) = \boldsymbol{SO}(2n$ or $2n+1)$ of rank n and let K be a closed connected subgroup of rank $n-1 > 0$. If K is odd, G has a subgroup B isomorphic to $(Z_2)^2$ with $B \cap \mathrm{ad}(G)K = 1$. If K is even and Γ is a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$, then every abelian subgroup of Γ is cyclic.*

Proof. Let $V = \{v_1, \cdots, v_n\}$ be the canonical parameter of K in G and let $b \in G$ have order 2. The eigenvalues of b are all 1 or -1. As $\det. b = 1$, the multiplicity of the eigenvalue -1 is some even number $2s$. It is clear that b is $\mathrm{ad}(G)$-conjugate to $\exp(\frac{1}{2}X_1 + \frac{1}{2}X_2 + \cdots + \frac{1}{2}X_s)$, where $\mathsf{X}_G = \{X_1, \cdots, X_n\}$ is our canonical integral basis, so the arithmetic criterion (Theorem 6) says that $b \in \mathrm{ad}(G)K$ if and only if some sum of s of the v_j, without repetitions, is even. When the v_j are all odd this means that

$b \in \mathrm{ad}(G)K$ if and only if s is even; when one of the v_j is even and $s < n$, this implies that $b \in \mathrm{ad}(G)K$.

Suppose K is odd. Then each of the v_j is odd, so we must exhibit a $(Z_2)^2$ in G in which every element $\neq 1$ has the eigenvalue -1 of multiplicity congruent to 2 modulo 4. Let $\boldsymbol{I}_t$ be the $t \times t$ identity matrix; then such a $(Z_2)^2$ is given by generators

$$b_1 = \mathrm{diag}\{-1, -1, 1, \boldsymbol{I}_{q-3}\}, \qquad b_2 = \mathrm{diag}\{1, -1, -1, \boldsymbol{I}_{q-3}\}$$

Suppose that K is even, so one of the v_j is even. By Theorem 5 we need only show that Γ contains no $(Z_2)^2$, so we must show that a $(Z_2)^2$ in G has an element $\neq 1$ with eigenvalue -1 of multiplicity not equal to $2n$. A $(Z_2)^m$ in $\boldsymbol{SO}(q)$ is conjugate to a group of diagonal matrices. It follows that G contains a $(Z_2)^2$ where every element $\neq 1$ has eigenvalue -1 with multiplicity $2n$ only if $q = 3$. That case was ruled out by the assumption rank. $K > 0$. *QED.*

III. 4. Elements of order 2 which act freely.

THEOREM 8. *Let G be a classical group $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, $\boldsymbol{SO}(2n)$ or $\boldsymbol{Spin}(2n)$ and let K be an even subgroup (hence closed and connected, and* rank. G − rank. $K = 1$*). Let Γ be a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$. Then Γ has at most one element of order 2, and an element of order 2 in Γ is central in G. Let H be a closed connected subgroup of $\boldsymbol{SU}(n)$ such that* rank. $\boldsymbol{SU}(n)$ − rank. $H = 1$ *and let Σ be a finite subgroup of $\boldsymbol{SU}(n)$ such that $\Sigma \cap \mathrm{ad}(\boldsymbol{SU}(n))H = 1$. Then both n and H are even if Σ has an element of order 2.*

Proof. Suppose $G \neq \boldsymbol{Spin}(2n)$ and let $\gamma \in \Gamma$ have order 2. As in the proof of Theorem 7, the arithmetic criterion shows that γ has the eigenvalue -1 with multiplicity $2n$ if $G = \boldsymbol{SO}(2n)$ or $\boldsymbol{Sp}(n)$, and with multiplicity n if $G = \boldsymbol{U}(n)$ or $\boldsymbol{SU}(n)$. Hence γ is conjugate to $-\boldsymbol{I}$, the negative of the identity matrix in G. As $-\boldsymbol{I}$ is central in G, $\gamma = -\boldsymbol{I}$ and is central in G.

Now suppose that $G = \boldsymbol{Spin}(2n)$ and $f\colon G \to \boldsymbol{SO}(2n)$ is the natural map. Let -1 denote the element of order 2 in ker. f. If -1 is in Γ or K, then Γ or K consists of whole f-fibres and we have $f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K) = \boldsymbol{I}_{2n}$. If -1 is in neither Γ nor K, then either $f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K) = \boldsymbol{I}_{2n}$ or Γ has an element $\gamma \neq 1$ such that $\boldsymbol{I}_{2n} \neq f(\gamma) \in f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K)$. We will show that this last alternative does not occur. For if it does, K has a conjugate K' such that $-\gamma \in K'$. γ has order 2, for $\gamma \notin \ker. f$ but $\gamma^2 = (-\gamma)^2 \in \Gamma \cap K'$. We can pass to a conjugate of γ and assume

$\gamma = e_1 \cdot e_2 \cdot \cdots \cdot e_{2s}$, where the e_j are an orthonormal basis of Euclidean space $\boldsymbol{R}^{2n}$, taken as generators of the Clifford algebra $\boldsymbol{C}(\boldsymbol{R}^{2n})$, and dots denote Clifford multiplication. If $s = n$, γ is central and thus, by Theorem 5, the only element of Γ of order 2. If $s < n$, let $\beta = e_{2s} \cdot e_{2s+1} \in \boldsymbol{Spin}(2n)$ and $\mathrm{ad}(\beta)\gamma = -\gamma$. This implies that both γ and $-\gamma$ are in $\mathrm{ad}(G)K$, which contradicts $\Gamma \cap \mathrm{ad}(G)K = 1$. Now we can assume that $f(\Gamma) \cap \mathrm{ad}(\boldsymbol{SO}(2n))f(K) = \boldsymbol{I}_{2n}$. $f(K)$ is an even subgroup of $\boldsymbol{SO}(2n)$ because K is even in $\boldsymbol{Spin}(2n)$, so an element of $f(\Gamma)$ of order 2 is $-\boldsymbol{I}_{2n}$. It follows that an element of Γ of order 2 lies in $f^{-1}(\{\pm \boldsymbol{I}_{2n}\})$, hence is central in $\boldsymbol{Spin}(2n)$. Uniqueness follows from Theorem 5.

Let $G = \boldsymbol{SU}(n)$ and let $\sigma \in \Sigma$ have order 2. If H is odd, the arithmetic criterion implies that the eigenvalue -1 of σ has odd multiplicity, contradicting $\det.\sigma = 1$. Thus H is even. If n is odd, we again contradict $\det.\sigma = 1$, for, H being even, the arithmetic criterion says that $\sigma = -\boldsymbol{I}_n$. *QED.*

Chapter IV. Finite subgroups of classical groups which have all abelian subgroups cyclic.

Theorems 5 and 7 tell us that if G is a classical group and K is a closed connected subgroup, assumed to be an even subgroup if G is special orthogonal, such that $\mathrm{rank}.\, G - \mathrm{rank}.\, K = 1$, and Γ is a finite subgroup of G such that $\Gamma \cap \mathrm{ad}(G)K = 1$, then every abelian subgroup of Γ is cyclic. For this reason, we'll examine the finite groups with all abelian subgroups cyclic.

IV. 1. Classification of finite groups with all abelian subgroups cyclic. The finite groups with all abelian subgroups cyclic fall into two classes ([14], Chapter I)—those with all Sylow subgroups cyclic, and those with odd Sylow subgroups cyclic and 2-Sylow subgroups generalized quaternionic. H. Zassenhaus ([16], p. 198, p. 202) and M. Suzuki ([13], p. 689) have given a complete classification of these groups in terms of generators and relations. We will not use this classification, but rather will rely on a simpler description given in H. Zassenhaus' book ([17], p. 175) for the finite groups with all Sylow subgroups cyclic, and on the fact that a finite group with all abelian subgroups cyclic has all Sylow subgroups cyclic if its 2-Sylow subgroups are not generalized quaternionic. For reference, the generalized quaternionic groups are the groups $\boldsymbol{Q}2^a$ of order 2^a, $a \geqq 3$, given by

$$A^{2^{a-1}} = 1, \quad B^2 = A^{2^{a-2}}, \quad BAB^{-1} = A^{-1}, \quad a \text{ integer}, \quad a \geqq 3.$$

A finite group of order N with all Sylow subgroups cyclic is given by $A^m = B^n = 1$, $BAB^{-1} = A^r$, $0 < m$, $mn = N$, $((r-1)n, m) = 1$, $r^n \equiv 1 \pmod{m}$.

Our plan of attack is to calculate representations of these groups in the classical groups, and find conditions under which the image of a representation acts freely on an appropriate coset space.

IV. 2. Classical representations of the generalized quaternionic groups. Following G. Vincent ([14], Chapter III), elementary techniques of representation theory tell us that the irreducible unitary representations of the generalized quaternionic group $\boldsymbol{Q}2^a$ are:

1. The 4 $\boldsymbol{U}(1)$-representations given by $A \to \pm 1$ and $B \to \pm 1$.
2. The $2^{a-2} - 1$ $\boldsymbol{U}(2)$-representations S_r, $1 \leqq r < 2^{a-2}$, given by

$$S_r\colon A \to \begin{pmatrix} u^r & 0 \\ 0 & u^{-r} \end{pmatrix} \text{ and } B \to \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}, \text{ where } u = \exp(2\pi i/2^{a-1}).$$

Note that S_r is faithful if and only if r is odd. Let S denote the $\boldsymbol{U}(1)$- representation $A \to 1$ and $B \to -1$.

It follows that a special unitary representation of $\boldsymbol{Q}2^a$ is an appropriate sum of $\boldsymbol{U}(1)$-representations plus a sum of some of

1. The 2^{a-3} $\boldsymbol{SU}(2)$-representations S_r, where r is odd.
2. The 2^{a-3} $\boldsymbol{SU}(3)$-representations $S_r + S$, where r is even.
3. The $\boldsymbol{SU}(4)$-representations $S_{r_1} + S_{r_2}$, where r_1 and r_2 are even.

Similarly, a symplectic representation of $\boldsymbol{Q}2^a$ is an appropriate sum of $\boldsymbol{U}(1)$-representations plus a sum of some of

1. The 2^{a-3} $\boldsymbol{Sp}(1)$-representations S_r, where r is odd.
2. The 2^{a-3} $\boldsymbol{Sp}(1)$-representations $S_r + S_r{}^*$, where r is even and $S_r{}^*$ is the complex conjugate representation of S_r.

A unitary, special unitary or symplectic representations of $Q2^a$ is faithful if and only if it has a summand S_r with r odd.

S_r is unitarily equivalent to its conjugate representation $S_r{}^*$, and is equivalent to a real representation if and only if r is even. As before, we set $R(t) = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \in \boldsymbol{SO}(2)$; the irreducible orthogonal representations of $\boldsymbol{Q}2^a$ are:

1. The 4 $\boldsymbol{O}(1)$-representations given by $A \to \pm 1$ and $B \to \pm 1$.

2. The $2^{a-3}-1$ $\boldsymbol{O}(2)$-representations S_r, r even, unitarily equivalent to the corresponding $\boldsymbol{U}(2)$-representations, given by $S_r\colon A \to R(r/2^{a-1})$ and $B \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. The 2^{a-3} $\boldsymbol{O}(4)$-representations T_r, r odd, equivalent to $S_r + S_r{}^*$, given by $T_r\colon A \to \begin{pmatrix} R(r/2^{a-1}) & 0 \\ 0 & R(-r/2^{a-1}) \end{pmatrix}$ and $B \to \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$.

A special orthogonal representation of $\boldsymbol{Q}2^a$ is an appropriate sum of $\boldsymbol{O}(1)$-representation plus a sum of some of

1. The $\boldsymbol{SO}(3)$-representations $S_r + S$ (r even, of course).

2. The $\boldsymbol{SO}(4)$-representations $S_{r_1} + S_{r_2}$ with r_1 and r_2 even.

3. The $\boldsymbol{SO}(4)$-representations T_r (r odd, of course).

An orthogonal or special orthogonal representation of $\boldsymbol{Q}2^a$ is faithful if and only if it has a summand T_r.

Each of the T_r can be lifted to a faithful $\boldsymbol{Spin}(4)$-representation of $\boldsymbol{Q}2^a$. Let T be one of the T_r and let $\{e_j\}$ be the orthonormal basis of $\boldsymbol{R}^4$ with respect to which our matrices are written; the $\{e_j\}$ generate the Clifford algebra $\boldsymbol{C}(\boldsymbol{R}^4)$. We choose $T'(A) \in \boldsymbol{Spin}(4)$ over $T(A)$ and $T'(B) \in \boldsymbol{Spin}(4)$ over $T(B)$. We then have

$$T'(A) = \pm(\cos x + e_2 \cdot e_1 \sin x) \cdot (\cos x - e_4 \cdot e_3 \sin x),$$

$$T'(A)^{-1} = \pm(\cos x - e_2 \cdot e_1 \sin x) \cdot (\cos x + e_4 \cdot e_3 \sin x)$$

and

$$T'(B) = \pm \tfrac{1}{2}(1 + e_3 \cdot e_1) \cdot (1 + e_4 \cdot e_2),$$

where dots denote Clifford multiplication, and $x = \pi r/2^{a-1}$. A short calculation shows that $T'(A)^{2^{a-2}} = e_1 \cdot e_2 \cdot e_4 \cdot e_3 = T'(B)^2$. Another calculation shows that $T'(B) \cdot T'(A) = T'(A)^{-1} \cdot T'(B)$. It follows that $T'(A)$ and $T'(B)$ generate a $\boldsymbol{Q}2^a$ in $\boldsymbol{Spin}(4)$, so T' extends to a $\boldsymbol{Spin}(4)$-representation of $\boldsymbol{Q}2^a$. T' is faithful because it covers a faithful representation.

Let $V = S_{r_1} + S_{r_2}$, $r_i = 2u_i$, a non-faithful $\boldsymbol{SO}(4)$-representation of $\boldsymbol{Q}2^a$. If $V'(A) \in \boldsymbol{Spin}(4)$ lies over $V(A)$ and $V'(B) \in \boldsymbol{Spin}(4)$ lies over $V(B)$, a short calculation shows that $V'(B)^2 = -1$, $V'(B) \cdot V'(A) \cdot V'(B)^{-1} = V'(A)^{-1}$, and $V'(A)^{2^{a-2}} = -1$ if and only if $u_1 + u_2$ is odd. In other words, V' extends to a $\boldsymbol{Spin}(4)$-representation $V'$of $\boldsymbol{Q}2^a$ if and only if one of the u_i is odd and the other is even. In that case, V' is faithful and -1 is the element of order 2 in $V'(\boldsymbol{Q}2^a)$.

Let $U = S_r + S$, $r = 2u$, a non-faithful $\boldsymbol{SO}(3)$-representation of $\boldsymbol{Q}2^a$.

Choosing $U'(A)$ and $U'(B)$ in $\boldsymbol{Spin}(3)$ over $U(A)$ and $U(B)$, we see that $U'(B)^2 = -1$, $U'(B) \cdot U'(A) \cdot U'(B)^{-1} = U'(A)^{-1}$, and $U'(A)^{2^{a-2}} = -1$ if and only if u is odd. Thus U' extends to a $\boldsymbol{Spin}(3)$-representation U' of $\boldsymbol{Q}_{2^a}$ if and only if u is odd; in that case, U' is faithful and -1 is the element of order 2 in $U'(\boldsymbol{Q}_{2^a})$.

IV. 3. Unitary representations of finite groups which have all Sylow subgroups cyclic. Let Γ be a finite group of order N with every Sylow subgroup cyclic. Γ is given by two generators A and B with relations $A^m = B^n = 1$, $BAB^{-1} = A^r$, $0 < m$, $mn = N$, $((r-1)n, m) = 1$, $r^n \equiv 1 \pmod{m}$. Note that m is odd; if m were even, r would be odd because A and A^r have the same order, so $2 \mid ((r-1)n, m)$, where we denote *a divides b* by $a \mid b$. Note also that not r but only the mod m residue class of r is important. Let ϕ be the Euler ϕ-function and let G_m be the multiplicative group of integers prime to m, taken modulo m. As m is odd, there can be no confusion with the exceptional Lie group $\boldsymbol{G}_2$. Given $C \in \Gamma$, let $\{C\}$ denote the cyclic subgroup of Γ generated by C. Let d be the order of r in G_m. As $r^n \equiv 1 \pmod{m}$, $d \mid n$ and we can write $n = n'd$. If $m_i \mid m$, set $d_i =$ order of r in G_{m_i}, $n = d_i n_i'$. G. Vincent has proved ([14], p. 156):

Γ has exactly $\phi(m_j)n'd/d_j^2$ irreducible unitary representations of degree d_j. On restricting one of these representations to $\{A\}$, it has kernel $\{A^{m_j}\}$. As m_j runs over all divisors of m, including 1 and m, these representations of degree d_j give all irreducible unitary representations of Γ.

This, together with other results of G. Vincent ([14], Chapter III) make it fairly easy to verify that the irreducible unitary representations of degree d_j of Γ are given by:

$$f_{jk}(A) = \operatorname{diag}\{\exp(2\pi i/m_j), \exp(2\pi ir/m_j), \cdots, \exp(2\pi ir^{d_j-1}/m_j)\}$$

$$f_{jk}(B) = \exp(2\pi ik/n) \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ for } 1 \leqq k \leqq \phi(m_j)n'd/d_j^2.$$

Chapter V. A second application of the arithmetic criterion to the classical groups.

V. 1. Generalized quaternionic subgroups of classical groups which act freely. Suppose G is a classical group of rank n and K is a closed

connected subgroup of rank $n-1$, assumed to be an even subgroup if G is a special orthogonal group. A generalized quaternionic subgroup Γ of G can be considered to be the image of a faithful G-representation of the appropriate $\boldsymbol{Q}2^a$. Since we know these representations explicitly, we can apply the arithmetic criterion to check whether $\Gamma \cap \mathrm{ad}(G)K = 1$. The importance of this procedure is that if no generalized quaternionic subgroup of G can act freely on G/K, then every finite subgroup of G that acts freely on G/K has all Sylow subgroups cyclic. Such groups have particularly simple structure among the finite groups with all abelian subgroups cyclic.

THEOREM 9.1. *Let G be a classical group $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, or $\boldsymbol{SO}(2n$ or $2n+1)$, K a closed connected subgroup with* rank. G — rank. $K = 1$, *$V = \{v_1, \cdots, v_n\}$ the canonical parameter of K in G, and q the number of v_j which are odd. Then G has a generalized quaternionic subgroup Γ such that $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if*

n is even and q is odd, if $G \neq \boldsymbol{Sp}(n)$; q is odd or $q = n$, if $G = \boldsymbol{Sp}(n)$. If G has a generalized quaternionic subgroup Γ such that $\Gamma \cap \mathrm{ad}(G)K = 1$ and if B is any generalized quaternionic group, then G has a subgroup B' isomorphic to B such that $B' \cap \mathrm{ad}(G)K = 1$. If K is an even subgroup in case $G = \boldsymbol{SO}(2n$ or $2n+1)$, if

n is odd or q is even, if $G \neq \boldsymbol{Sp}(n)$; q is even and $q < n$, if $G = \boldsymbol{Sp}(n)$, and if Σ is a finite subgroup of G such that $\Sigma \cap \mathrm{ad}(G)K = 1$, then every Sylow subgroup of Σ is cyclic.

Proof. Let Γ be a generalized quaternionic subgroup of $G = \boldsymbol{U}(n)$, $\boldsymbol{Sp}(n)$ or $\boldsymbol{SO}(2n$ or $2n+1)$, considered as the image of a faithful G-representation F of $\boldsymbol{Q}2^a$. Checking the various possible summands of F, we see that every element of Γ is $\mathrm{ad}(G)$-conjugate to a power of $F(A)$ or $F(B)$; it follows that $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $\{F(A)\} \cap \mathrm{ad}(G)K = 1 = \{F(B)\} \cap \mathrm{ad}(G)K$. We will apply the arithmetic criterion (Theorem 6) to these two cyclic subgroups of Γ. When we do this, the integer m in the formulation of the arithmetic criterion will be a power of 2, as Γ is a 2-group, so we may ignore the integers b_j which are even. As the kernel of a non-faithful G-representation of $\boldsymbol{Q}2^a$ contains the element $B^2 = A^{2^{a-2}}$ of order 2, this means we need only consider the faithful summands of F. Replacing Γ by an $\mathrm{ad}(G)$-conjugate if necessary, we can assume $F = S_{r_1} + \cdots + S_{r_s}$ if $G = \boldsymbol{U}(n)$ or $\boldsymbol{Sp}(n)$ and $F = T_{r_1} + \cdots + T_{r_s}$ if $G = \boldsymbol{SO}(2n$ or $2n+1)$, where the r_j are odd. An application of the arithmetic criterion now tells us that the condtion for $\Gamma \cap \mathrm{ad}(G)\mathrm{K} = 1$ is that, for every element g of the

group $\boldsymbol{S}(n)$ of all permutations on $\{1, 2, \cdots, n\}$, $\sum_j r_j(v_{g(2j-1)} - v_{g(2j)})$ and $\sum_j(v_{g(2j-1)} - v_{g(2j)})$ are odd if $G = \boldsymbol{U}(n)$, $\sum_j r_j(\pm v_{g(j)})$ and $\sum_j(\pm v_{g(j)})$ are odd for any arrangement of $\pm$ signs if $G = \boldsymbol{Sp}(n)$, and

$$\sum_j r_j(\pm v_{g(2j-1)} - (\pm v_{g(2j)})) \text{ and } \sum_j(\pm v_{g(2j-1)} - (\pm v_{g(2j)}))$$

are odd for any arrangement of $\pm$ signs, requiring only that the number of minus signs be even if $G = \boldsymbol{SO}(2n)$, if $G = \boldsymbol{SO}(2n \text{ or } 2n+1)$. As the r_j are odd, the first number has the same residue mod 2 as the second, so we may ignore the first one in each case. Similarly, we may ignore signs. We now see that the condition for $\Gamma \cap \mathrm{ad}(G)K = 1$ is that, for every $g \in \boldsymbol{S}(n)$, $\sum_{j=1}^{2s} v_{g(j)}$ is odd if $G \neq \boldsymbol{Sp}(n)$; $\sum_{j=1}^{s} v_{g(j)}$ is odd if $G = \boldsymbol{Sp}(n)$. This is independent of a and of the r_j, and depends only on n, s and q. It happens if and only if $n = 2s$ and q is odd if $G \neq \boldsymbol{Sp}(n)$; q is odd or $q = n$ if $G = \boldsymbol{Sp}(n)$.

The theorem is now proved except for $G = \boldsymbol{SU}(n)$. Suppose $G = \boldsymbol{SU}(n) \subset \boldsymbol{U}(n)$ and let K' be a closed connected subgroup of rank $n-1$ in $\boldsymbol{U}(n)$ such that $K = K' \cap \boldsymbol{SU}(n)$. If Γ is a subgroup of $\boldsymbol{SU}(n)$, $\Gamma \cap \mathrm{ad}(\boldsymbol{SU}(n))K = \Gamma \cap \mathrm{ad}(\boldsymbol{U}(n))K'$, so we are done because a generalized quaternionic subgroup of $\boldsymbol{U}(n)$ can be assumed, for purposes of checking $\Gamma \cap \mathrm{ad}(\boldsymbol{U}(n))K' = 1$, to be the image of a sum of S_{r_j} with r_j odd, and hence can be assumed to lie in $\boldsymbol{SU}(n)$. *QED.*

The situation with the ***Spin*** groups is more complicated because of the relative abundance of faithful ***Spin***-representations of the generalized quaternionic groups.

LEMMA 9.1. *Let K be a closed connected subgroup of rank $n-1 > 0$ in $G = \boldsymbol{Spin}(2n \text{ or } 2n+1)$ and let -1 be the element of order 2 in the kernel of the natural projection f: $\boldsymbol{Spin} \to \boldsymbol{SO}$. Then K is an even subgroup of G if and only if it contains -1.*

Proof. K contains -1 if and only if -1 lies on a 1-parameter subgroup of K, as K is connected. Let Y be a 1-parameter subgroup of K. As -1 is central in G we may assume that $(f\,Y)(t) = \exp(\sum_s t a_s X_s)$, where the exponential is taken in $\boldsymbol{SO}(2n \text{ or } 2n+1)$ and $\{X_1, \cdots, X_n\}$ is our canonical integral basis for $\boldsymbol{SO}(2n \text{ or } 2n+1)$. We may assume Y normalized so that $(f \cdot Y)(t) = \boldsymbol{I}$ if and only if t is an integer; the a_s are then integers and -1 lies on Y if and only if $-1 = Y(1)$. $Y(1) = -1$ if and only if the number of odd a_j is odd.

Let $V = \{v_1, \cdots, v_n\}$ be the canonical parameter of K in G, i.e., the canonical parameter of $f(K)$ in $\boldsymbol{SO}$. If K is odd, $\sum_s v_s a_s = 0$ implies that

an even number of a_s are odd and $-1 \neq Y(1)$. If K is even, we can assume v_1 even and v_2 odd; we construct a conjugate X of a 1-parameter subgroup of K which contains -1 by $(f \cdot X)(t) = \exp(t(v_2X_1 - v_1X_2))$. *QED.*

We mention an interesting consequence of Lemma 9.1:

COROLLARY 9.1. *Let K be a closed connected subgroup of rank $n-1>0$ in $G = \boldsymbol{SO}(2n \text{ or } 2n+1)$. Then G/K is simply connected if and only if K is an even subgroup of G; $\pi_1(G/K) = Z_2$ if K is odd.*

Proof. The universal covering of G/K is $\boldsymbol{Spin}(2n \text{ or } 2n+1)/K'$, where K' is the identity component of $f^{-1}(K)$. *QED.*

THEOREM 9.2. *Let K be a closed connected subgroup of rank $n-1>0$ in $G = \boldsymbol{Spin}(2n \text{ or } 2n+1)$, $V = \{v_1, \cdots, v_n\}$ the canonical parameter of K in G, and q the number of v_j which are odd.*

Suppose K is even. Then G has a generalized quaternionic subgroup Γ such that $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if n is even (say $n = 2s$) and q is odd, and any such Γ is $\mathrm{ad}(G)$-conjugate to the image of a faithful G-representation F' of a $\boldsymbol{Q}^{2a}$, where $F' = T'_{r_1} + \cdots + T'_{r_s}$ for some choice of odd integers r_j. If $n = 2s$, q is odd, $a \geqq 3$, $\{r_1, \cdots, r_s\}$ are odd integers and

$$\Gamma = (T'_{r_1} + \cdots + T'_{r_s})(\boldsymbol{Q}^{2a'}),$$

then $\Gamma \cap \mathrm{ad}(G)K = 1$. If n is odd or q is even, and Σ is a finite subgroup of G such that $\Sigma \cap \mathrm{ad}(G)K = 1$, then every Sylow subgroup of Σ is cyclic.

Suppose K is odd. Given $a \geqq 3$, G has a subgroup Γ isomorphic to $\boldsymbol{Q}^{2^a}$ such that $\Gamma \cap \mathrm{ad}(G)K = 1$, -1 is the element of order 2 in Γ, $f(\Gamma)$ is a dihedral 2-subgroup of $\boldsymbol{SO}(2n \text{ or } 2n+1) = G'$ such that $f(\Gamma) \cap \mathrm{ad}(G')f(K) = 1$ and $f(\Gamma)$ is $\mathrm{ad}(G')$-conjugate to the image of a non-faithful G'-representation F of $\boldsymbol{Q}^{2^a}$ which is a sum of representations of the forms $S_{2r} + S_{4s}$, r odd, and $S_{2t} + S$, t odd.

Proof. $f\colon G = \boldsymbol{Spin} \to \boldsymbol{SO} = G'$ being the natural projection, -1 is the element of order 2 in $\ker. f$ and $K' = f(K)$. Suppose K is even, so $-1 \in K$; given a subgroup Γ of G, $\Gamma \cap \mathrm{ad}(G)K = 1$ if and only if $f(\Gamma) \cap \mathrm{ad}(G')K' = 1$ and $-1 \notin \Gamma$. A generalized quaternionic subgroup Γ of G not containing -1 is $\mathrm{ad}(G)$-conjugate to the image of a faithful G-representation $F' = T'_{r_1} + \cdots + T'_{r_s} + t'$, t' not faithful, of $\boldsymbol{Q}^{2^a}$. $f \cdot F'$ is a faithful G'-representation $F = T_{r_1} + \cdots + T_{r_s} + t$, t not faithful, of $\boldsymbol{Q}^{2^a}$. In the proof of Theorem 9.1 we saw that $f(\Gamma) \cap \mathrm{ad}(G')K' = 1$ if and only if $n = 2s$ and q is odd, and this is independent of the choices of a and of the odd integers r_j.

If $n=2s$, then t, and hence t', is trivial because F represents by matrices of determinant 1.

Now suppose K is odd. Given $a \geqq 3$ we will construct a subgroup Γ' of G', isomorphic to the dihedral group $\boldsymbol{D}2^{a-1}$ of 2^{a-1} elements, such that $\Gamma' \cap \mathrm{ad}(G')K' = 1$ and $\Gamma = f^{-1}(\Gamma')$ is isomorphic to $\boldsymbol{Q}2^a$. Then -1 will be the element of order 2 in Γ. Given $\gamma \in \Gamma \cap \mathrm{ad}(G)K$, $f(\gamma)=1$; as $-1 \notin K$ it follows that $\gamma = 1$, so $\Gamma \cap \mathrm{ad}(G)K = 1$.

Let Γ' be a dihedral 2-subgroup of G'. As $\boldsymbol{D}2^{a-1}$ is the quotient of $\boldsymbol{Q}2^a$ by the subgroup generated by $B^2 = A^{2^{a-2}}$, we may view Γ' as the image of a representation $F = S_{2r_1} + \cdots + S_{2r_u} + S_{4t_1} + \cdots + S_{4t_v} + s$, r_j odd and s a sum of $\boldsymbol{O}(1)$-representations, of $\boldsymbol{Q}2^a$. Let $2w'$ and w be the multiplicities of the eigenvalue -1 of $s(A)$ and $s(B)$; the eigenvalue -1 of $F(B)$ has multiplicity $u+v+w=2x$. K' is odd because K is odd; as in the proof of Theorem 9.1 the arithmetic criterion shows that $\{F(A)\} \cap \mathrm{ad}(G')K' = 1$ if and only if u is odd when $a > 3$, if and only if $u + w'$ is odd when $a = 3$. It also shows that $\{F(B)\} \cap \mathrm{ad}(G')K' = 1$ if and only if x is odd.

The representations F which lift to ***Spin*** are of the form

$$(*) \qquad \begin{aligned} F = &(S_{2r_1} + S_{4s_1}) + \cdots + (S_{2r_p} + S_{4s_p}) \\ &+ (S_{2t_1} + S) + \cdots + (S_{2t_q} + S) \end{aligned}$$

with r_j, t_j odd and where $S: A \to 1$, $B \to -1 \in \boldsymbol{O}(1)$. In this case, every element of Γ' is conjugate to a power of $F(A)$ or $F(B)$ and we have $u = p+q = x$, $w' = 0$; it follows that $\Gamma' \cap \mathrm{ad}(G')K' = 1$ if and only if $p+q$ is odd. rank. $K > 0$ implies $n > 1$, so we can find non-negative integers p and q with $p+q$ odd and $4p+3q \leqq 2n$, hence a G'-representation F of the form (*) with $p+q$ odd. *QED.*

V. 2. Subgroups of the unitary group which have all Sylow subgroups cyclic and act freely. Suppose that K is a closed connected subgroup of $\boldsymbol{U}(n)$ of rank $n-1$ and Γ is a finite subgroup of $\boldsymbol{U}(n)$ with all Sylow subgroups cyclic. Γ is conjugate to the image of a faithful representation of an abstract finite group Σ with all Sylow subgroups cyclic, and we can replace Γ by that conjugate. We will apply our arithmetic criterion (Theorem 6) to see whether $\Gamma \cap \mathrm{ad}(\boldsymbol{U}(n))K = 1$. This is of considerable interest if n is odd or the number of odd elements of the canonical parameter of K in $\boldsymbol{U}(n)$ is even, for then every finite subgroup B of $\boldsymbol{U}(n)$ such that $B \cap \mathrm{ad}(\boldsymbol{U}(n))K = 1$ has every Sylow subgroup cyclic.

Let N be the order of Σ. We represent Σ by generators and relations:

$A^m = B^n = 1$, $BAB^{-1} = A^r$, $0 < m$, $mn = N$, $((r-1)n, m) = 1$ and $r^n \equiv 1 \pmod m$. If Σ is cyclic, $m = 1$ and this becomes $B^N = 1$.

If Γ is cyclic of order t and has a generator γ with eigenvalues

$$\exp(\pi i r_1/t), \cdots, \exp(2\pi i r_n/t),$$

and $V = \{v_1, \cdots, v_n\}$ is the canonical parameter of K in $\boldsymbol{U}(n)$, a direct application of the arithmetic criterion shows that $\Gamma \cap \mathrm{ad}(\boldsymbol{U}(n))K = 1$ if and only if $\sum_s r_s v_{g(s)}$ is prime to t for every element g of the permutation group $\boldsymbol{S}(n)$. We will, then, ignore this case and henceforth assume that Σ is not cyclic.

In the notation of § IV. 3, we can assume that Γ is the image of the faithful representation $F = \sum_{j=1}^{a} \sum_{p=1}^{b_j} f_{jk_jp}$ of Σ in $\boldsymbol{U}(q)$.

Given an integer u and a divisor m_j of m, we define $u^{(j)} = (u, d_j)$, $0 \leqq u_j < d_j$ and $u_j \equiv u \pmod{d_j}$, $d_j^{(u)} \cdot u^{(j)} = d_j$, and

$$r^{(u)} = 1 + r^{u_j} + r^{2u_j} + \cdots + r^{(d_j^{(u)}-1)u_j}.$$

Given a second integer v, we define $h(u, v)$ to be the order of $B^u A^v$ in Σ. A calculation shows that $f_{jk_jp}(B^u A^v)$ has eigenvalues

$$\exp(2\pi i[(h(u,v)/nm_j)(k_{jp}um_j + u^{(j)}m_jn_j'e + n_j'vu^{(j)}r^{(u)}r^t)]/h(u,v))$$

for $0 \leqq e < d_j^{(u)}$ and $0 \leqq t < u^{(j)}$. If u is prime to d_j, a calculation shows that this means that $f_{jk_jp}(B^u A^v)$ has eigenvalues $\exp(2\pi i[k_{jp}u - en_j']/n)$ for $0 \leqq e < d_j$, hence is $\mathrm{ad}(\boldsymbol{U}(d_j))$-conjugate to $f_{jk_jp}(B^u)$. If $d \mid u$, another calculation shows that the eigenvalues of $f_{jk_jp}(B^u A^v)$ can be written, on setting $u = wd$ so $u/n = w/n'$, as

$$\exp(2\pi i[k_{jp}wm + (mn'/m_j)vr^t]/mn') \text{ for } 0 \leqq t < d_j.$$

Now set $N_{jpe} = \sum_{c<j} d_c b_c + (p-1)d_j + e + 1$, and, given an integer u, set $N_{jpet}(u) = \sum_{c<j} d_c b_c + (p-1)d_j + eu^{(j)} + t + 1$. With this notation, an application of the arithmetic criterion now yields:

THEOREM 10. *Let K be a closed connected subgroup of rank $q-1$ in the unitary group $\boldsymbol{U}(q)$, $V = \{v_1, \cdots, v_q\}$ the canonical parameter of K in $\boldsymbol{U}(q)$, $\boldsymbol{S}(q)$ the group of all permutations on $\{1, 2, \cdots, q\}$ and Γ a finite subgroup of $\boldsymbol{U}(q)$ which is the image of a faithful representation $F = \sum_{j=1}^{a} \sum_{p=1}^{b_j} f_{jk_jp}$ of the abstract finite non-cyclic group Σ with all Sylow subgroups cyclic. Then $\Gamma \cap \mathrm{ad}(\boldsymbol{U}(q))K = 1$ if and only if for every $g \in \boldsymbol{S}(q)$ we have:*

1. $\sum_{j=1}^{a}\sum_{p=1}^{b_j}\sum_{e=0}^{d_j-1} v_{g(N_{jpe})}\cdot(k_{jp}+enj')$ *is prime to* n.

2. $\sum_{j=1}^{a}\sum_{p=1}^{b_j}\sum_{e=0}^{d_j-1} v_{g(N_{jpe})}\cdot(mk_{jp}+(m/m_j)n'r^e)$ *is prime to* mn'.

3. Given integers u and v with $1\leqq u<n$, $1\leqq v<m$ *and* $1<(u,d)<d$,

$$\sum_{j=1}^{a}\sum_{p=1}^{b_j}\sum_{e=0}^{d_j^{(u)}-1}\sum_{t=0}^{u^{(j)}-1} v_{g(N_{jpet(u)})}\cdot(h(u,v)/nm_j)\cdot(k_{jp}um_j+u^{(j)}m_jn_j'e+n_j'vu^{(j)}r^{(u)}r^t)\not\equiv 0 \pmod{h(u,v)}.$$

To adapt these formulae to the other classical groups, we proceed as follows:

$\boldsymbol{SU}(q)$. K must have rank $q-2$ and Γ must lie in $\boldsymbol{SU}(q)$. Formulae (1, 2, 3; Theorem 10) remain unchanged.

$\boldsymbol{Sp}(q)$. $\Gamma\subset\boldsymbol{Sp}(q)\subset\boldsymbol{U}(2q)$, $\boldsymbol{S}(q)$ must be replaced by the group $\boldsymbol{S}'(q)$ of all signed permutations on V, and, for each formula of Theorem 10, the numbers following the v's fall into two sets, one of which is the negative of the other, and only one must be summed.

$\boldsymbol{SO}(2q+1)$. $\Gamma\subset\boldsymbol{SO}(2q+1)\subset\boldsymbol{U}(2q+1)$ and we proceed as for $\boldsymbol{Sp}(q)$.

$\boldsymbol{SO}(2q)$. $\Gamma\subset\boldsymbol{SO}(2q)\subset\boldsymbol{U}(2q)$, $\boldsymbol{S}(q)$ must be replaced by the group $\boldsymbol{S}''(q)$ of all signed permutations on V which involve an even number of changes of sign, and we proceed as for $\boldsymbol{Sp}(q)$.

$\boldsymbol{Spin}(2q$ or $2q+1)$. We proceed as for $\boldsymbol{SO}(2q$ or $2q+1)$.

Recall that if $G=\boldsymbol{U}(q)$, $\boldsymbol{SU}(q)$, $\boldsymbol{Sp}(q)$, $\boldsymbol{SO}(2q$ or $2q+1)$, or $\boldsymbol{Spin}(2q$ or $2q+1)$, and if $K=\boldsymbol{U}(q-1)$, $\boldsymbol{SU}(q-1)$, $\boldsymbol{Sp}(q-1)$, $\boldsymbol{SO}(2q-1$ or $2q-2)$, or $\boldsymbol{Spin}(2q-1$ or $2q-2)$, respectively, imbedded in the usual way, the canonical parameter of K in G is $\{1,0,\cdots,0\}$. With this in mind, we can use Theorem 10 to generalize some rather nice theorems of H. Zassenhaus [16] and G. Vincent [14]:

COROLLARY 10.1. *Let G be a classical group $\boldsymbol{U}(q)$, $\boldsymbol{SU}(q)$, $\boldsymbol{Sp}(q)$, $\boldsymbol{SO}(2q$ or $2q+1)$ or $\boldsymbol{Spin}(2q$ or $2q+1)$ and let K be a closed connected subgroup such that* rank. $G-$ rank. $K=1$ *and the canonical parameter of K in G is $\{1,0,0,\cdots,0\}$. Let Γ be a finite subgroup of G with $\Gamma\cap\mathrm{ad}(G)K=1$, such that the order of Γ is either the product of two primes or is prime to $2q$. Then Γ is cyclic.*

Proof. Suppose first that the order of Γ is prime to $2q$. As every abelian subgroup of Γ is cyclic and Γ has odd order, every Sylow subgroup of Γ is cyclic. Formula 2 of Theorem 10 now says that $mk_{jp} + (m/m_j)n'r^e$ is prime to mn', hence to m, so $m_j = m$ and consequently each $d_j = d$. This implies that d divides both q and the order of Γ, which are relatively prime, so $d = 1$. But $d = 1$ ifplies $r = 1$ and thus that Γ is cyclic.

Suppose Γ has order mn with m and n prime, and that Γ is not cyclic. As Γ is not abelian, $m \neq n$. It follows that every Sylow subgroup of Γ is cyclic, so we look at Theorem 10, which, by switching m and n if necessary, is directly applicable. $d \mid n$ and n is prime, so $d = 1$ or $d = n$. As $d = 1$ implies that Γ is cyclic, $d = n$. Formula 2 of Theorem 10 then shows $m_j = m$, so $n_j' = n' = 1$. Formula 1 of Theorem 10 then says that $k_{jp} + e$ is prime to n for $0 \leqq e < n$, which is impossible. *QED.*

In addition to providing known information on spheres, Corollary 10.1 tells us something about the Grassmann manifolds $\boldsymbol{SO}(2q)/\boldsymbol{SO}(2q-2)$, $\boldsymbol{SO}(2q+1)/\boldsymbol{SO}(2q-1)$ and $\boldsymbol{SO}(2q+1)/\boldsymbol{SO}(2q-2)$. The formulae of Theorem 10 can yield all sorts of information by placing special conditions on the canonical parameter.

Chapter VI. Manifolds with non-zero Euler characteristic.

After stating that we would for the most part concentrate on the case rank. G — rank. $K \leqq 1$, we devoted our attention primarily to the case rank. G — rank. $K = 1$. In this chapter, we will prove a theorem about the case where rank. $G =$ rank. K. First recall the well-known fact ([10], p. 15) that a coset space G/K of a compact connected Lie group G by a closed subgroup K has Euler characteristic $\chi(G/K) \geqq 0$, and that $\chi(G/K) > 0$ if and only if rank. $G =$ rank. K. We will prove:

THEOREM 11. *Let M be a compact connected Riemannian homogeneous manifold with Euler characteristic $\chi(M) \neq 0$. Then there are only a finite number, up to isometry, of Riemannian manifolds which admit M as a Riemannian covering manifold.*

Remark. If M' admits a Riemannian covering of multiplicity n by M, we have $\chi(M) = n \cdot \chi(M')$. As $\chi(M')$ must be an integer, it is clear, intuitively speaking, that one can go down only a finite number of steps from M. The theorem says, then, that there are only a finite number of steps from M. The theorem says, then, that there are only a finite number of "directions" in which one can go down. These various "directions" will

be seen to correspond roughly to the subgroups of the finite group G/G_0, where G_0 is the identity component of the group G of isometries of M.

Proof. We will first show that we may assume M simply connected, so that we will only have to consider normal coverings, i. e., coverings which are effectuated by the group of deck transformations. Let K_0 be the intersection of an isotropy subgroup K of the group G of isometries of M with the identity component G_0 of G, so $M = G_0/K_0$. K_0 contains a maximal torus of G_0 but contains no nontrivial normal subgroup of G_0; it follows that G_0 is centerless, hence semisimple, and thus has finite fundamental group. The homotopy sequence of the fibring $G_0 \to G_0/K_0 = M$ then shows that M has finite fundamental group, so the universal Riemannian covering manifold M'' is compact. We will be done if we show that only a finite number, up to isometry, of Riemannian manifolds admit a Riemannian covering by M'', so we may assume M simply connected.

We now need only show that there are only a finite number of properly discontinuous subgroups of G which give mutually non-isometric quotient manifolds of M. As conjugate subgroups of G give isometric quotients, we need only show that there are only a finite number of mutually non-conjugate properly discontinuous subgroups of G. As rank. G = rank. K, $\mathrm{ad}(G)K$ contains G_0 so a properly discontinuous subgroup of G meets G_0 only at 1 and is thus isomorphic to a subgroup of the finite group G/G_0. The proof of Theorem 11 is thus reduced to:

LEMMA 11.1 (MOSTOW). *Let G be a compact Lie group and Γ a finite group. Then G contains only a finite number of conjugacy classes of isomorphs of Γ.*

Proof. Suppose the contrary and let $\{\Gamma_n\}$ be a sequence of mutually non-conjugate isomorphs of Γ which lie in G. We can assume that $\Gamma_n = \{\gamma_{1n}, \gamma_{2n}, \cdots, \gamma_{kn}\}$ ordered so that $\gamma_{jn} \to \gamma_{jm}$ is an isomorphism $\Gamma_n \to \Gamma_m$ for every m and n. As Γ is finite and G compact, we can assume that each sequence $\{\gamma_{jn}\}_n$ converges, $\{\gamma_{jn}\} \to \gamma_j$. It is clear that $\Sigma = \{\gamma_1, \cdots, \gamma_k\}$ is a subgroup of G, although we don't yet know that the γ_j are all distinct. A theorem of D. Montgomery and L. Zippin ([9], p. 216) says that Σ has a neighborhood U such that every subgroup of G in U is $\mathrm{ad}(G)$-conjugate to a subgroup of Σ. As the Γ_n eventually lie in U, this contradicts their mutual non-conjugacy. *QED.*

The proof of Theorem 11 also furnishes a proof of:

THEOREM 11′. *Let M be a compact connected Riemannian homogeneous*

manifold. If M has finite fundamental group, there are only a finite number, up to isometry, of Riemannian manifolds with a given fundamental group which admit M as a Riemannian covering manifold. In any given case, there are only a finite number, up to isometry, of Riemannian manifolds which admit a normal Riemannian covering by M with a given group of deck transformations.

THE UNIVERSITY OF CHICAGO.

REFERENCES.

[1] A. Borel, "Fixed point theorems for elementary abelian groups, I," *Seminar on Transformation Groups*, notes, The Institute for Advanced Study, Princeton, 1959.

[2] ———, "Commutative subgroups and torsion in compact Lie groups," (to appear).

[3] ——— and J, de Siebenthal, "Les sous-groupes fermés de rang maximum des groupes de Lie clos," *Commentarii Mathematici Helvetici*, vol. 23 (1949), pp. 200-221.

[4] C. Chevalley, *Theory of Lie groups*, vol. I, Princeton, 1946.

[5] P. E. Conner, "On the action of a finite group on $S^n \times S^n$," *Annals of Mathematics* (2), vol. 66 (1957), pp. 586-588.

[6] A. Heller, "A note on spaces with operators," *Illinois Journal of Mathematics*, vol. 3 (1959), pp. 98-99.

[7] A. G. Kurosh, *The theory of groups*, vol. I, New York, 1955.

[8] J. Milnor, "Groups which act on S^n without fixed points," *American Journal of Mathematics*, vol. 79 (1957), pp. 623-630.

[9] D. Montgomery and L. Zippin, *Topological transformation groups*, New York, 1955.

[10] H. Samelson, "On curvature and characteristic of homogeneous spaces," *Michigan Mathematical Journal*, vol. 5 (1958), pp. 13-18.

[11] P. A. Smith, "Permutable periodic transformations," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 30 (1944), pp. 105-108.

[12] N. Steenrod, *The topology of fibre bundles*, Princeton, 1951.

[13] M. Suzuki, "On finite groups with cyclic Sylow subgroups for all odd primes," *American Journal of Mathematics*, vol. 77 (1955), pp. 657-691.

[14] G. Vincent, "Les groupes linéaires finis sans points fixes," *Commentarii Mathematici Helvetici*, vol. 20 (1947), pp. 117-171.

[15] B. L. van der Waerden, *Modern Algebra*, vol. II, New York, 1950.

[16] H. Zassenhaus, "Über endliche Fastkörper," *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, bd. 11 (1935), pp. 187-220.

[17] ———, *The theory of groups*, second edition, New York, 1958.

FUNDAMENTAL GROUPS OF COMPACT SOLVMANIFOLDS.*

By Louis Auslander.[1]

Introduction. Let S be a connected, simply connected solvable Lie group and let G be a closed subgroup of S. Then the homogeneous space S/G is called a solvmanifold. In the special case where the solvable Lie group is nilpotent, the homogeneous space is called a nilmanifold. The study of nilmanifolds was initiated by Malcev, [1], who succeeded in giving a very detailed account of the structure of such spaces. Recently, much effort has been put into extending the Malcev results to the class of solvmanifolds. The major contributions in this direction have been obtained by Mostow [2] and Wang [4]. However, there still remained open at the completion of this work the following two questions.

Problem 1. Give an algebraic characterization of the fundamental groups of *compact* solvmanifolds.

Problem 2. Give an algebraic characterization of the discrete uniform subgroups of solvable Lie groups.

Now clearly every discrete uniform subgroup is a fundamental group of a compact solvmanifold, but an example of Wang [4] shows that the converse is not true. The purpose of this paper is to prove the following theorem:

Theorem I. *Exery fundamental group of a solvmanifold is the fundamental group of a compact solvmanifold.*

Since an algebraic characterization of the fundamental groups of solvmanifolds is contained in [4], this gives a complete solution to problem 1. Problem 2 still remains open and, in the author's opinion is of considerable interest.

* Received March 1, 1959; revised October 1, 1959.

[1] This paper was in preparation during part of the time the author was being supported under a National Science Foundation Grant.

Part I.

1. Recapitulation. Let Γ be the fundamental group of a solvmanifold. Then Γ can be imbedded as a discrete uniform subgroup of a simply connected solvable Lie group S, where S has at most a finite number of components (see [4]). Further, from the construction of S, it follows that the identity component of S, which we will denote by S_0, contains the commutator subgroup of S. But by [3], $S = F \cdot S_0$, where F is a finite group and the dot denotes the semi-direct product. Hence F is an abelian group. Now $\Gamma_0 = \Gamma \cap S_0$ is a discrete uniform subgroup of S_0 and Γ_0 is of finite index in Γ. If N is the maximal normal analytic nilpotent subgroup of S_0, then, again by the construction of S, F induces the identity transformation on S_0/N. Further, $N_0 = N \cap \Gamma$ is a discrete uniform subgroup of N and Γ_0/N_0 is a free abelian group on, we may assume, s generators. Again by the construction given in [4], if N_0 and $\theta_1, \cdots, \theta_s$ generate Γ, then there exist positive integers λ_i, $i = 1, \cdots, s$, such that $(\theta_i)^{\lambda_i}$ lie on one parameter subgroups in S_0.

2. Existence theorem. Let $\mathcal{S}_0$ be any simply connected, connected solvable Lie group for which there exists a homomorphism h of $\mathcal{S}_0$ onto S_0; then $h^{-1}(\Gamma_0)$ is a uniform subgroup of $\mathcal{S}_0$ and $\mathcal{S}_0/h^{-1}(\Gamma_0)$ is homeomorphic to S_0/Γ_0 (see [2]). Further, if $\mathcal{N}_0$ is the maximal normal analytic nilpotent subgroup of $\mathcal{S}_0$, then the kernel K of h is contained in $\mathcal{N}_0$ and $h^{-1}(\Gamma_0) \cap \mathcal{N}_0$ is a uniform subgroup of $\mathcal{N}_0$ ([2], prop. 1, p. 22).

Definition. Let F be a group of automorphisms of S_0. We will say that F can be lifted to $\mathcal{S}_0$ if there exists a group F^*, F^* isomorphic to F, of automorphisms of $\mathcal{S}_0$ which map K onto itself and induce the automorphism group F on S_0. If for S_0, F there exists $\mathcal{S}_0$ and F^*, as above, we will say that $\mathcal{S}_0$, F^* is an enlargement of S_0, F.

THEOREM 1. *Let Γ be the fundamental group of a solvmanifold and let $\Gamma \subset S$, where S satisfies the conditions of* §1. *Assume that there exists an enlargement $\mathcal{S}_0$, F^* of S_0, F. Then if F^* can be imbedded in a torus group T of automorphisms of $\mathcal{S}_0$, such that T acts trivially on $\mathcal{S}_0/\mathcal{N}_0$, Γ is the fundamental group of a* compact *solvmanifold.*

Proof. Consider $\mathcal{S} = F^* \cdot \mathcal{S}_0$. Then h can be extended to a homomorphism h_* of $\mathcal{S}$ onto S if we define h_* by

$$h_*(f^*, s) = (i(f^*), h(s)),$$

where $f^* \in F^*$, $i(f^*)$ is the image of f^* in F, $s \in \mathcal{S}_0$ and $h(s) \in S$. There

the kernel of h_* equals the kernel of h or K and hence the kernel of h_* is connected, simply connected and contained in $\mathfrak{N}_0$, where $\mathfrak{N}_0$ is the maximal normal analytic nilpotent subgroup of $\mathfrak{H}_0$. Consider in $h_*^{-1}(\Gamma)$ pre-images $\eta_1, \cdots, \eta_s$ of $\theta_1, \cdots, \theta_s$ in Γ. Since $h_*^{-1}(\Gamma)/h_*^{-1}(\Gamma) \cap \mathfrak{N}_0$ is free abelian on s generators, $h^{-1}(\Gamma) \cap \mathfrak{N}_0$ and $\eta_1, \cdots, \eta_s$ generate $h_*^{-1}(\Gamma)$.

We will use the symbol $L(\ )$ to denote the Lie algebra of the Lie group in the parentheses. Now let $x_i \in L(S_0)$ be such that $\exp(x_i) = \theta_i^{\lambda_i}$. Since h maps $\mathfrak{H}_0$ onto S_0, there exists $X_i \in L(\mathfrak{H}_0)$ such that X_i maps onto x_i by the induced mapping of $L(\mathfrak{H}_0)$ onto $L(S_0)$. Since $h(\exp(X_i)) = \theta_i^{\lambda_i}$, we must have that $\exp(X_i) \equiv \eta_i^{\lambda_i} \bmod K$. Now T is a compact group, T acts trivially on $L(\mathfrak{H}_0/\mathfrak{N}_0)$, and $X_i \notin L(\mathfrak{N}_0)$. Hence we can find $Y_i \in L(\mathfrak{H}_0)$ such that Y_i is invariant under T and $Y_i \equiv X_i \bmod L(\mathfrak{N}_0)$. Let $\eta_i = (f_i, s_i)$. Since $\mathfrak{N}_0$ is a characteristic subgroup in $\mathfrak{H}_0$, it is normal in $\mathfrak{H}$ and we can consider $U: S \to \mathfrak{H}/\mathfrak{N}_0$. Then $\mathfrak{H}/\mathfrak{N}_0 = F^* + V^s$, where V^s is an s dimensional vector space and $+$ denotes the direct sum. Let $\exp((1/\lambda_i)Y_i) = (0, r_i)$. Let $U(\eta_i) = (f_i, \xi_i)$. Then $U(\eta_i^{\lambda_i}) = (f_i^{\lambda_i}, \lambda_i\xi_i) = (0, \lambda_i\xi_i)$. But $U(\exp(\xi_i Y_i)) = U(0, r_i^{\lambda_i}) = (0, \lambda_i\xi_i)$. Hence $U(\exp((1/\lambda_i)Y_i)) = (0, \xi_i)$. This shows that $f^*_i((1/\lambda_i)Y_i) \equiv \eta_i \bmod \mathfrak{N}_0$.

Now let $S^\# = T \cdot \mathfrak{H}_0$, where the dot denotes the semi-direct product. Consider the subalgebra A of $L(\mathfrak{H}^\#)$ spanned by $L(\mathfrak{N}_0)$ and $(t_i + (1/\lambda_i)Y_i)$, where $t_i \in L(T)$ such that $\exp(t_i) = f^*_i$. Let $\mathfrak{H}^*$ be the connected subgroup whose Lie algebra is A. We wish to show that $\mathfrak{H}^*$ is simply connected and closed in $\mathfrak{H}^\#$. Now $\mathfrak{H}^* \supset \mathfrak{N}_0$ and $\mathfrak{H}^*/\mathfrak{N}_0$ is a vector subgroup of the abelian Lie group $T \cdot \mathfrak{H}_0/\mathfrak{N}_0$. Since both $\mathfrak{H}^*/\mathfrak{N}_0$ and $\mathfrak{N}_0$ are closed and simply connected, the same is true of $\mathfrak{H}^*$. Hence $p^{-1}(\Gamma) \cap \mathfrak{H}^*$ is a closed subgroup of $\mathfrak{H}^*$, since each is a closed subset of $\mathfrak{H}^\#$. But $\mathfrak{H}^* \supset p^{-1}(\Gamma)$. For $\mathfrak{H}^* \supset \mathfrak{N}_0$ and $\mathfrak{H}^*$ contains $\exp(t_i + (1/\lambda_i)Y_i)$. But

$$\exp(t_i + (1/\lambda_i)Y_i) = \exp(t_i)\exp((1/\lambda_i)Y_i)$$

since $[t_i, Y_i] = 0$. Hence $\exp(t_i + (1/\lambda_i)Y_i) \equiv \eta_i \bmod \mathfrak{N}_0$. Hence $\mathfrak{H}^* \supset \eta_i$, $i = 1, \cdots, s$. This shows that $\mathfrak{H}^* \supset h_*^{-1}(\Gamma)$. It remains to verify that $\mathfrak{H}^*/h_*^{-1}(\Gamma)$ is compact. This follows easily from the fact that $h_*^{-1}(\Gamma) \cap \mathfrak{N}_0$ is uniform in $\mathfrak{N}_0$ and the fact that in $\mathfrak{H}^*/\mathfrak{N}_0$ the images of $\eta_1, \cdots, \eta_s$ generate a discrete uniform subgroup. This proves that $\mathfrak{H}^*/h_*^{-1}(\Gamma)$ can be represented as a fiber bundle with compact fiber and base. Hence $\mathfrak{H}^*/h_*^{-1}(\Gamma)$ is compact and our theorem is proved.

3. A decomposition of $L(S_0)$. Let $L(S_0)$ denote the Lie algebra of S_0. Then, since F is a finite abelian group, and F acts trivially on $L(S_0/N_0)$,

where $L(N_0)$ is invariant under all automorphisms of $L(S_0)$, there is a direct sum decomposition of $L(S_0)$

$$L(S_0) = L(N_0) + C,$$

where F acts trivially on C. However, since F is abelian, $L(N_0)$ can be divided into the direct sum of invariant two dimensional spaces $V^2{}_\alpha$ $\alpha = 1, \cdots, r$, and one dimensional spaces $V^1{}_\beta$, $\beta = 1, \cdots, t$. We will consider each $V^2{}_\alpha$ as oriented and given an inner product with respect to which F acts as orthogonal transformations. Now, we may define an equivalence relations on the objects $V^2{}_\alpha$ and divide them into equivalence classes $\mathfrak{U}^2{}_i$, $i = 1, \cdots, r'$, as follows:

$V^2{}_\alpha$ is said to be equivalent to $V^2{}_{\alpha'}$ or $V^2{}_\alpha \sim V^2{}_{\alpha'}$ if either

1. For all $f \in F$, f induces the same positive angle of rotation in $V^2{}_\alpha$ and $V^2{}_{\alpha'}$,

or

2. For all $f \in F$, f induces the negative of the angle of rotation in $V^2{}_\alpha$ as in $V^2{}_{\alpha'}$.

It is clear that in each equivalence class $\mathfrak{U}^2{}_i$ each member can be so oriented that condition 1 will always hold. We will for the rest of this paper assume that this has been done.

LEMMA 1. *Let $V^2{}_\alpha \in \mathfrak{U}^2{}_i$, let $X \in C$ and let $[X, V^2{}_\alpha]$ denote the linear mapping of the linear space spanned by $V^2{}_\alpha$ under bracket with X. Then either*

a) *$[X, V^2{}_\alpha]$ is a linear transformation of $V^2{}_\alpha$ onto $V^2{}_{\alpha'} \in \mathfrak{U}^2{}_i$ (i, the same as in the hypothesis) or* b) *$[X, V^2{}_\alpha]$ maps $V^2{}_\alpha$ into zero.*

Proof. Let (Y, Z) be an oriented orthonormal basis for $V^2{}_\alpha$. Then consider the linear space spanned by $[X, Y]$ and $[X, Z]$. Then there exists $f \in F$ such that

$$f(Y) = \cos\theta Y + \sin\theta Z,$$
$$f(Z) = -\sin\theta Y + \cos\theta Z,$$

where $|\cos\theta| < 1$ and $0 \leqq \theta \leqq 2\pi$. But

$$f[X, Y] = \cos\theta[X, Y] + \sin\theta[X, Z],$$
$$f[X, Z] = -\sin\theta[X, Y] + \cos\theta[X, Z].$$

Now if $[X, Y] = 0$, then $[X, Z]$ must be zero. For otherwise

$$f[X, Z] = \cos\theta[X, Z]$$

where $|\cos\theta| \neq 1$. Since F is a finite group, this is impossible. An analogous argument shows that if $[X, Z] = 0$, then $[X, Y] = 0$, also. Hence the image of $V^2{}_\alpha$ under $[X, V^2{}_\alpha]$ is a 2 dimensional invariant space or it is the null space. But if the image of $V^2{}_\alpha$ under $[X, V^2{}_\alpha]$ is a 2 dimensional invariant space this space must also be in $\mathfrak{U}^2{}_i$. This proves the lemma.

COROLLARY. *Let $V^2{}_\alpha$ and $V^2{}_\beta$ be given and let h_1 and h_2 be two linear mappings of $V^2{}_\alpha$ onto $V^2{}_\beta$ induced by a succession of bracketing with elements $X \in C$. Then if $V^2{}_\alpha$ is oriented, the orientations of $V^2{}_\beta$ induced by h_1 and h_2 are compatible.*

Proof. Let (Y, Z) be oriented bases for $V^2{}_\alpha$ and let the linear transformation f operate on $V^2{}_\beta$ with $0 < |\cos\theta| < 1$. Then relative to the bases $(h_1(Y), h_1(Z))$ and $(h_2(Y), h_2(Z))$ the linear transformation f has exactly the same matrix. This implies that the orientation of $V^2{}_\beta$ induced by h_1 and h_2 are compatible.

By the above corollary we may orient coherently all elements of $\mathfrak{U}^2{}_i$ with the property that one gets mapped onto another by an element of C.

Since each $V^2{}_\alpha \in \mathfrak{U}^2{}_i$ has an inner product and an orientation assigned to it, we may consider the one parameter group of orientation preserving rotations. Let T_i denote this one parameter group of orientation preserving rotations of the vector spaces in $\mathfrak{U}^2{}_i$. Then on $V^2{}_\alpha$, T_i will be defined explicitly by:

$$T_i(Y_\alpha) = \cos 2\pi t Y_\alpha + \sin 2\pi t Z_\alpha,$$

$$T_i(Z_\alpha) = -\sin 2\pi t Y_\alpha + \cos 2\pi t Z_\alpha,$$

$0 \leqq t \leqq 1$. Note that for any other compatibly oriented orthonormal basis for $V^2{}_\alpha$ the form of the equations expressing the action of T_i would be identical.

LEMMA 2. *Let $V^2{}_\alpha \in \mathfrak{U}^2{}_i$, $X \in C$ and $t \in T_i$. Then*

$$t[X, Y] = [X, tY]$$

for $Y \in V^2{}_\alpha$.

Proof. If the image of $V^2{}_\alpha$ under bracket with X is zero, the lemma is trivially true. Hence assume this is not the case and that (Y_α, Z_α) is an oriented orthonormal basis of $V^2{}_\alpha$. Then $([X, Y_\alpha], [X, Z_\alpha])$ will be oriented orthonormal basis of the image of $V^2{}_\alpha$ under $[X, V^2{}_\alpha]$. Hence T_i induces

the same linear transformations relative to the basis $([X, Y_\alpha], [X, Z_\alpha])$ of the image of $V^2{}_\alpha$ as it does relative to the basis (Y_α, Z_α) for $V^2{}_\alpha$. This proves the lemma.

4. Construction of enlargements. It is now our purpose to show that the hypothesis of Theorem 1 can always be satisfied. We will restrict ourselves to the obvious Lie algebra formulation of the hypothesis and show that this can always be satisfied. Since any automorphism group of a Lie algebra induces an isomorphic automorphism group of any associated simply connected Lie group and every algebra has a simply connected, connected associated Lie group, once we have shown that the hypothesis can be satisfied for Lie algebras, this will prove that this is also the case for Lie groups. The proof that the hypothesis of the existence theorem can always be satisfied will be arrived at in two stages. Step one will consist in showing that there exists an *infinite* dimensional Lie algebra with certain properties. Step two will consist in finding a finite homomorphic image of the infinite Lie algebra which will satisfy the hypothesis of Theorem 1.

Let $w_1, \cdots, w_k$ be any finite set of symbols. Then we may form the free Lie algebra over the reals, $L(k)$, consisting essentially of the module with the non-associative words in $w_1, \cdots, w_k$ as basis divided by the ideal of the "proper" relations. We will now make this more precise.

Let W be the set of non-associative words in the letters $w_1, \cdots, w_k$ form the module $M(W)$ of linear combinations over the reals of the elements of W. Now define the bilinear mapping of $M(W) \times M(W) \to M(W)$ which extends the mapping which maps the pair of words (u, w) into the word $u(w)$. In this algebra, consider the ideal I generated by the relations of the form

1. $x \cdot x = 0$

2. $(x \cdot (y \cdot z)) + (y \cdot (z \cdot x)) + (z \cdot (x \cdot y)) = 0$

for $x, y, z \in M(W)$. Then $L(k)$ is defined as $M(W)/I$. We will call $L(k)$ the fee Lie algebra on k generators. $L(k)$ is clearly an infinite dimensional vector space over the reals.

LEMMA 3. *Let $V(W)$ be the sublinear space of $M(W)$ spanned by the elements $w_1, \cdots, w_k$. Then every non-singular linear mapping A of $V(W)$ onto itself is uniquely extendable to an automorphism A^* of $L(k)$ onto itself.*

Proof. Let $W_1, \cdots, W_k$ and $w_1, \cdots, w_k$ be two sets of k symbols. Then

the free Lie algebra generated by each of these sets of symbols are clearly isomorphic under the mapping which sends every word in the lower case letters into the corresponding word with capital letters. Now let $A(w_i), \cdots, A(w_k)$ be a new basis for $V(W)$. Then call $A(w_i) = W_i$. Then the isomorphism described above determines an automorphism of $L(k)$ onto itself.

LEMMA 4. *Any k dimensional Lie algebra is the homomorphic image of $L(k)$.*

This lemma is obvious.

5. Infinite enlargements. In $L(S_0)$ we have already discussed the sets $V^2{}_\alpha$, $V^1{}_\beta$ and C. Let I be the linear subspace of $L(S_0)$ spanned by vectors of $V^1{}_\beta$ which are invariant under all elements of F. Let V be the subspace spanned by the remaining elements of $V^1{}_\beta$. Then $V^2{}_\alpha$, I, V and C form a direct sum decomposition of $L(S_0)$, where $V^2{}_\alpha$, I, V form a direct sum decomposition of $L(N_0)$. Let (Y_α, Z_α) be an oriented basis for $V^2{}_\alpha$, as discussed in § 3, let $X_1{}^*, \cdots, X_s{}^*$ be a basis for I. Let U be a finite dimensional vector space isomorphic to V with a fixed isomorphism ϕ chosen in advance. Let $U_1, \cdots, U_k$ be the images of $W_1, \cdots, W_k$ under ϕ. Let L be the free Lie algebra generated by the symbols Y, Z, X^*, W, X and U under the appropriate ranges of subscripts. Then we can define a homomorphism of L onto $L(S_0)$ by mapping all U into zero and every other letter into the corresponding vector in $L(S_0)$. Call this homomorphism h.

In order to define an enlargement, we have to define an automorphism group F^* acting on L. But our Lemma 3, showed this was equivalent to defining a group of automorphisms of the linear space L spanned by the symbols Y, Z, etc. Now there is a natural isomorphism of the subspace spanned by all the symbols except U onto $L(S_0)$. Use this isomorphism to define F^* on this subspace of L. Complete the definition of F^* as follows: Define

$$f^* U_i = \operatorname{sign}(f^*, W_i) U_i,$$

where $\operatorname{sign}(f^*, W_i)$ is defined by the equation

$$f W_i = \operatorname{sign}(f^*, W_i) W_i.$$

This determines the group F^* of automorphism of L. F^* preserves the kernel of the homomorphism h and induces F on $L(S_0)$, since

$$\begin{array}{ccc} L & \xrightarrow{f^*} & L \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ L(S_0) & \xrightarrow{f} & L(S_0) \end{array}$$

is a commutative diagram.

LEMMA 5. *The automorphism group F^* can be imbedded in a torus group T^* of automorphism of L.*

Proof. Define the circle group T_i^* for L: Define it on the vector space spanned by the letters Y_α, Z_α, for $V^2{}_\alpha \in \mathfrak{U}^2{}_i$ as in § 3, define it to act trivially on all other letters. This defines T_i^* on L by Lemma 3. Define a circle group T on L as follows:

$$T(W_A) = \cos 2\pi t W_A + \sin 2\pi t U_A,$$
$$T(U_A) = -\sin 2\pi t W_A + \cos 2\pi t U_A,$$

$A = 1, \cdots, k$, $0 \leqq t \leqq 1$ and T acts trivially on all other symbols. Consider the torus group $\mathfrak{J}$ generated by the circle groups T_i^* and T^*. $\mathfrak{J}$ clearly contains F^* as a subgroup. This proves the lemma.

6. Finite enlargements. We are now faced with the problem of finding an $\mathfrak{S}$ "between" L and S_0 which is finite dimensional and such that the kernel of the homomorphism which maps L onto S is invariant under $\mathfrak{J}$. This will then induce a torus group T on $\mathfrak{S}$ and the pair $\mathfrak{S}$, T will satisfy the hypothesis of Theorem 1. This will then prove Theorem I stated in the introduction.

Let $L(N_0)$ be ξ step nilpotent. Consider in L the ideal I^* generated by the following words.

1. All words of length $\xi + 1$ not containing any letters X.
2. If in $L(S_0)$ $[X_i, X_j] = \sum a_{ijs} X^*{}_s$, then the word $[X_i, X_j] - \sum a_{ijs} X^*{}_s$ in L.
3. If in $L(S_0)$ $[X_i, W_A] = \sum a_{iAB} W_B$, then the word $[X_i, W_A] - \sum a_{iAB} W_B$ in L.
4. If in $L(S_0)$ $[X_i, W_A] = \sum a_{iAB} W_B$, then the word $[X_i, U_A] - \sum a_{iAB} U_B$ in L.

5. If in $L(S_0)$ $[X_i, Y_\alpha] = Y_{\alpha'}$, then the word
$[X_i, Y_\alpha] - Y_{\alpha'}$ in L.

6. If in $L(S_0)$ $[X_i, Z_\alpha] = Z_{\alpha'}$, then the word
$[X_i, Z_\alpha] - Z_{\alpha'}$ in L.

LEMMA 6. *$\mathfrak{d} = L/I^*$ is a finite dimensional solvable Lie algebra and h induces a homomorphism of $\mathfrak{d}$ into $L(S_0)$. Further, I^* is invariant under $\mathfrak{J}$ in L.*

Proof. To see that $\mathfrak{d}$ is finite dimensional and solvable, we note that $[\mathfrak{d}, \mathfrak{d}]$ consists of the images of words involving no X. But since all words not invloving X of length greater than $\xi + 1$ map into zero in $\mathfrak{d}$, we see that $[\mathfrak{d}, \mathfrak{d}]$ is a finite dimensional nilpotent Lie algebra. Further, $\mathfrak{d}/[\mathfrak{d}, \mathfrak{d}]$ is isomorphic to C and hence of finite dimension. This shows that $\mathfrak{d}$ is a finite dimensional solvable Lie algebra. To see that h induces a homomorphism of $\mathfrak{d}$ onto $L(S_0)$, we have only to note that I^* is certainly in the kernel of h.

It remains to show that I^* is invariant under $\mathfrak{J}$. We have only to show that I^* is invariant under the one parameter groups T^*_i and T^*. Clearly, if we see that the vector space spanned by the generators of I^* is invariant under these groups, then I^* is invariant under these groups. But by Lemma 2 this vector space is invariant under the groups T^*_i. To verify that this is also true for T^* is trivial. This proves the lemma.

This completes the proof of Theorem I.

REFERENCES.

[1] A. I. Malcev, "On a class of homogeneous spaces," *American Mathematical Society, Translations*, No. 39.

[2] G. D. Mostow, "Factor spaces of solvable groups," *Annals of Mathematics*, vol. 60 (1954), pp. 1-27.

[3] ———, "Self-adjoint group," *ibid.*, vol. 62 (1955), pp. 44-55.

[4] H. C. Wang, "Discrete subgroups of solvable Lie groups," *ibid.*, vol. 64 (1956), pp. 1-19.

COMPACT TRANSFORMATION GROUPS OF S^n WITH AN $(n-1)$-DIMENSIONAL ORBIT.*

By HSIEN-CHUNG WANG.[1]

1. Introduction. The compact and connected transformation groups of an n-sphere S^n with an $(n-1)$-dimensional orbit were first studied by Montgomery and Zippin [10]. When G has a fixed point, they determined the orbits and reduced this problem to the problem of compact, transitive transformation groups of homotopy spheres. Recently, under weaker conditions on the dimension of the orbits but assuming differentiability, Montgomery-Samelson-Yang show that G must act on S^n orthogonally. Later, Poncet [17] proved this without differentiability condition. When G has no fixed point, the situation is different and some results have been obtained by Mostert [18] and Nagano [19]. It is the aim of the present paper to classify these transformation groups when the action is differentiable. The main results can be stated as follows:

"Let G be a connected, compact and effective differentiable transformation group of S^n with an $(n-1)$-dimensional orbit. Suppose that either n is even and different from 4, or is odd and greater than 31. Then G acts on S^n orthogonally, and up to an equivalence in the sense of transformation groups, G falls into five main classes. If a subgroup of the rotation group $O(n+1)$ is regarded as a transformation group of the unit sphere in R^{n+1}, then these classes can be described as follows: (i) *G is a subgroup of $O(n+1)$ leaving invariant and transitive on the unit sphere S^{n-1} in a hyperplane of R^{n+1}.* (ii) *Let G', G'' be subgroups of $O(p+1)$, $O(q+1)$ transitive on the unit spheres S^p, S^q in R^{p+1}, R^{q+1} respectively, where $p>0$, $q>0$, $p+q=n-1$, and denote by $G'\times G''$ the direct product in the sense of linear groups.*[2] *G is a subgroup of $O(n+1)$ containing $G'\times G''$ as a normal sub-*

* Received September 24, 1959.

[1] This work was partially supported by the National Science Foundation.

[2] Let G' be a group of automorphisms of R^a, and G'' a group of automorphisms of R^b. By the direct product $G'\times G''$ in the sense of linear group, we mean the group $G'\times G''$ acting on the direct sum $R^a\oplus R^b$ in the natural manner.

group. (iii) *G is the tensor product $Sp(r) \otimes Sp(2)$ over the quaternions*[3] *regarded as a subgroup of $O(8r)$, $n = 8r - 1$, where $Sp(m)$ denotes the group of all $(m \times m)$ quaternion symplectic matrices.* (iv) *G is either the tensor product $SU(r) \otimes SU(2)$ or $U(r) \otimes U(2)$ regarded as subgroups of $O(4r)$, $n = 4r - 1$, where $U(m)$ denotes the unitary group and $SU(m)$ the unimodular unitary group.* (v) *G is the tensor product $O(p+2) \otimes O(2)$ regarded as a subgroup of $O(2p+4)$, $n = 2p + 3$.*"

We note that in Cases (i) and (ii), the classification of G is reduced to the classification of compact transitive transformation groups of spheres which is known.

Our proof consists of two parts. In the first part, we establish some general properties of the orbits and isotropic subgroups. In fact, from the results of Montgomery-Zippin, all the $(n-1)$-dimensional orbits X are homeomorphic, and besides, there are two orbits X_1, X_2 of lower dimension. Let L. L_1, L_2 be the isotropic subgroups corresponding to X, X_1, X_2 respectively. Then $X = G/L$, $X_1 = G/L_1$, $X_2 = G/L_2$. We can choose the isotropic subgroups so that $L \subset L_1$, $L \subset L_2$. From these two inclusions, two projections $f_1 : X \to X_1$, $f_2 : X \to X_2$ are defined. Let C_i be the mapping cylinder associated to f_i $(i = 1, 2)$. Then S^n can be written as the union $C_1 \cup C_2$, where $C_1 \cap C_2 = X$. By using this decomposition, it is proved that f_1 and f_2 have the following properties:[4]

(A) X is a sphere bundle over X_i with f_i as the projection $(i = 1, 2)$.

(B) Let f_i^* be the homomorphism of the cohomology with any coefficient induced by f_i $(i = 1, 2)$. Then f_i^* is injective, and, for $0 < m < n - 1$,

$$f_1^*(H^m(X_1)) + f_2^*(H^m(X_2)) = H^m(X), \qquad f_1^*(H^m(X_1)) \cap f_2^*(H^m(X_2)) = 0.$$

(C) Let $f_i^{\#}$ be the homomorphism of the fundamental groups induced by f_i. Then $f_1^{\#}(\ker f_2^{\#}) = \pi_1(X_1)$, $f_2^{\#}(\ker f_1^{\#}) = \pi_1(X_2)$.

If the differentiability is not assumed, (B) and (C) are still valid, but we can only prove that X is a homology sphere bundle over X_i.

We note that (B) and (C) reflect the global properties of S^n while (A)

[3] Let Q^r be the r-dimensional right quaternion vector space, and Q^2 the left quaternion plane. In the suitable manner, $Sp(r)$ and $Sp(2)$ can be regarded as the groups of symplectic transformations of Q^r and Q^2 respectively. Thus $Sp(r) \times Sp(2)$ acts naturally on the tensor product $Q^r \otimes Q^2$ which is a real vector space R^{8r}, and the action is real-linear. This group of automorphisms of R^{8r} is called the tensor product $Sp(r) \otimes Sp(2)$.

[4] Properties (A) and (B) have been proved independently by Mostert and Nagano.

is concerned with certain local properties. In fact, we have a partial converse. Let X, X_1, X_2 be compact manifolds and X orientable. Suppose $f_i: X \to X_i$ to be a fibre map with fibre F_i of positive dimension $(i = 1, 2)$. Denote by C_i the mapping cylinder associated with f_i, and by Σ the union of C_1 and C_2 with their lower bases X identified. Then condition (B) implies that Σ is a homology sphere while (C) implies that Σ is simply-connected. Since X is a manifold, Σ is locally euclidean at any point in $\Sigma - (X_1 \cup X_2)$. At points in X_i, the local homology depends on the homology of the fibre F_i.

The second part consists in the determination of the transformation group G. Taking account of the properties (A), (B), (C) and using known results on the real cohomology of homogeneous spaces, we can classify the system $\{G, L, L_1, L_2\}$ when $n > 31$. On the other hand, it can be proved that the action of G is completely determined by the triple $\{L, L_1, L_2\}$ of subgroups. In this way, the transformation groups $\{G, S^n\}$ are determined.

The author wishes to express his sincere thanks to Dr. G. E. Bredon and Professor H. Samelson for their valuable comments.

2. Mapping cylinder. For later use, we shall establish in this section a property about mapping cylinders. Let X, Y be two spaces, and $f: X \to Y$ a continuous surjective map. Suppose that any set V in Y is open if and only if $f^{-1}(V)$ is open in X. We define the mapping cylinder C associated with f to be the space obtained from the product $X \times I$ $(I = \{t: 0 \leqq t \leqq 1\})$ by identifying the points $(x, 1)$ and $(x', 1)$ when $f(x) = f(x')$. In the natural manner, C can be written as the union $X \cup (X \times (0,1)) \cup Y$, where $(0,1) = \{t: 0 < t < 1\}$, and X, Y retain their own topologies. We shall always take X and Y as closed subsets of C in this sense. X and Y will be called the lower and upper bases of C respectively. The map $r: C \to C$ defined by $r(x) = f(x)$, $r(y) = y$, $r((x,t)) = f(x)$, $x \in X$, $y \in Y$, $0 < t < 1$, is a retraction and is homotopic to the identity.

Let B be a subset of Y and $A = f^{-1}(B)$. Then the set

$$C_A = A \cup (A \times (0,1)) \cup B$$

is the mapping cyliner associated to $f|_A: A \to B$. We shall call C_A the sub-cylinder of C over A.

From the definition, it follows directly that

(2.1) *Let C be the mapping cylinder associated to a map $f: X \to Y$, and U a neighborhood in C of a point y of Y. If $f^{-1}(y)$ is compact, then there exists a real number η, $0 < \eta < 1$, such that $f^{-1}(y) \times [1-\eta, 1) \subset U$, where $[1-\eta, 1) = \{t: 1-\eta \leqq t < 1\}$.*

Now we can prove the following lemma:

(2.2) *Let $f: X \to Y$ be a fibre map with fibre F, where Y, F are compact manifolds, not necessarily connected, and C the mapping cylinder associated to f. If $C - X$ is a differentiable manifold and the imbedding of Y in $C - X$ is differentiable, then F is a homotopy sphere.*

Proof. Let $\dim Y = m$, $\dim F = r$. Then $\dim X = m + r$, and $\dim C = m + r + 1$. Take an open m-cell b^m in Y, and set $A = f^{-1}(b^m)$. Then A is the product bundle over b^m with fibre F. Now consider the subcylinder C_A of C over A. We know that $C_A = A \cup (A \times (0,1)) \cup b^m$, and that $C_A - A$ is an $(m+r+1)$-dimensional differentiable manifold containing b^m as a differentiable submanifold. Let y be a point of b^m. There exists a neighborhood U of y in C_A such that the difference set $U - b^m$ is homeomorphic with the product $R^{m+1} \times S^r$, where S^r denotes the r-sphere and R^{m+1} the $(m+1)$-dimensional euclidean space. By (2.1), we can find η, $0 < \eta < 1$, with the property that $F \times [1-\eta, 1) \subset U$, where $F = f^{-1}(y)$ is a fibre. Evidently

$$F \times [1-\eta, 1) \subset U - b^m \subset C_A - b^m.$$

Since $C_A - b^m$ is homeomorphic with $F \times b^m \times [0,1)$ and b^m is contractible, $F \times [1-\eta, 1)$ is a deformation retract of $C_A - b^m$, and hence the inclusion maps

$$\pi_i(F \times [1-\eta, 1)) \to \pi_i(C_A - b^m), \qquad i = 0, 1, 2, \cdots$$

are injective. From the fact that $U - b^m$ is homeomorphic with $R^{m+1} \times S^r$, it follows that $\pi_i(F) = \pi_i(F \times [1-\eta, 1)) = 0$ for $0 \leqq i \leqq r-1$, and that $\pi_i(F)$ has no torsion. By assumption, F is a compact manifold, and therefore F must be a homotopy sphere of dimension r.

3. Transformation groups of S^n with an $(n-1)$-dimensional orbit. Let G be a connected and compact group acting on S^n almost effectively with an $(n-1)$-dimensional orbit. Montgomery-Zippin [10, 11] proved that G is a Lie group, and that, with two exceptions, all the orbits under G are homeomorphic to one another. Moreover, there exists an arc A in S^n such that (i) each orbit intersects A in exactly one point, (ii) the isotropic subgroups of G at inner points of the arc A are all identical, and (iii) the orbits at the end points of A have dimension less than $n-1$. Now choose an inner point a of the arc A. Let a_1, a_2 be the end points of A, and $X = G(a)$, $X_1 = G(a_1)$, $X_2 = G(a_2)$ be the respective orbits. Denote by L, L_1, L_2 the isotropic subgroups of G at a, a_1, a_2 respectively. Then $L \subset L_1$, $L \subset L_2$,

$X = G/L$, $X_1 = G/L_1$, $X_2 = G/L_2$. Since G is a Lie group, X, X_1, X_2 are manifolds. The map $f_i: X \to X_i$ defined by $f_i(gL) = gL_i$, $g \in G$, is a fibre map with L_i/L as the fibre $(i = 1, 2)$.

Let us write A as the union of two arcs A_1 and A_2, where $A_1 = \widehat{a_1 a}$, $A_2 = \widehat{a_2 a}$. For each i, the orbit $G(A_i)$ of A_i under G is evidently the mapping cylinder C_i associated to the map $f_i: X \to X_i$. Thus we can write

$$S^n = C_1 \cup C_2, \qquad C_1 \cap C_2 = X. \tag{3.1}$$

From now on, let us assume that G acts on S^n differentiably. It follows that X_i is a submanifold imbedded in S^n, and hence in $C_i - X$, in the differentiable manner. (2.2) tells us then that L_i/L must be a homotopy sphere. From a theorem of Borel[5] [2], we know that L_i/L is actually a sphere $(i = 1, 2)$.

For each i, let $k_i: X \to C_i$ be the injection map, and $r_i: C_i \to X_i$ be the natural retraction of the mapping cylinder onto its upper base. Evidently

$$f_1 = r_1 k_1, \qquad f_2 = r_2 k_2. \tag{3.2}$$

Moreover, it is well-known that the homomorphism

$$r_i^*: H^*(X_i) \to H^*(C_i) \tag{3.3}$$

of the cohomology rings induced by the retraction r_i is bijective. Now we apply the Vietoris-Mayer sequence to (3.1). It follows at once that the homomorphism $H^s(C_1) + H^s(C_2) \to H^s(X)$ defined by $(v_1, v_2) \to k_1^*(v_1) - k_2^*(v_2)$, $v_1 \in H^s(C_1)$, $v_2 \in H^s(C_2)$, is bijective when $0 < s < n-1$. This fact together with (3.2) and (3.3) tells us that the homomorphism $H^s(X_1) + H^s(X_2) \to H^s(X)$ given by $(u_1, u_2) \to f_1^*(u_1) - f_2^*(u_2)$, $u_1 \in H^s(X_1)$, $u_2 \in H^s(X_2)$, is bijective for $0 < s < n-1$.

Summarizing the above results, we have

(3.4) *Let G be a connected and compact group acting differentiably on S^n with an $(n-1)$ dimensional orbit. Denote by X an $(n-1)$-dimensional orbit, and by X_1, X_2 the two particular orbits. There are natural maps $f_1: X \to X_1$, $f_2: X \to X_2$ which are sphere fibre maps. Let $f_i^*: H^*(X_i) \to H^*(X)$ denote the homomorphism of the cohomology rings with arbitrary coefficient ring $(i = 1, 2)$. Then f_i^* is injective, and, for $0 < s < n-1$,*

$$f_1^*(H^s(X_1)) + f_2^*(H^s(X_2)) = H^s(X), \qquad f^*(H^s(X_1)) \cap f_2^*(H^s(X_2)) = 0.$$

[5] When n is even, this follows from known results on homogeneous spaces with non-vanishing Euler characteristic. For arbitrary n, Borel's method [1] of determining the compact transitive transformation groups of spheres gives a definite way to prove this theorem, however, detail was not given. Matsushima has also shown this when n is odd and not too small. Poncet [17] gave the proof and provided the detail for the exceptional cases.

Now suppose X_i to be orientable. Since X is orientable, $\{X, X_i, f_i\}$ is an orientable sphere bundle [7], and hence we have the Gysin-Chern-Spanier sequence with integer coefficients

$$H^0(X_i) \to H^{m+1}(X_i) \xrightarrow{f_i^*} H^{m_i+1}(X) \to \cdots,$$

where $m_i = \dim X - \dim X_i = \dim L_i - \dim L$. Since f_i^* is injective, the characteristic class of the sphere bundle always vanishes. Therefore,

(3.5) *If X_i is orientable, then the sphere bundle $\{X, X_i, f_i\}$ has zero characteristic class (integer coefficients). Moveover, X and $X_i \times S^{m_i}$ have the same Poincaré polynomial.*

Since we always have the Gysin-Chern-Spanier sequence with integers mod 2 as the coefficient field, then

(3.6) *X and $X_i \times S^{m_i}$ have the same Poincaré polynomial if the coefficient field is of characteristic 2.*

4. Fundamental group of orbits. In this section, some properties of the fundamental group of the orbits will be established by using a theorem of van Kampen. We first digress a little and prove a proposition on free product of groups.

(4.1) *Let A, B_1, B_2 be finitely generated abstract groups, and $h_i: A \to B_i$ be surjective homomorphisms with kernel K_i $(i = 1, 2)$. Let N be the least normal subgroup of the free product $B_1 \square B_2$ containing the set $\{h_1(a)h_2(a^{-1}) : a \in A\}$ of elements in $B_1 \square B_2$. Then $A/K_1K_2 \approx (B_1 \square B_2)/N$.*

Proof. For any m symbols $x_1, \cdots, x_m$ and any set $\{P_\alpha\}$ of words in m variables, let us denote by $[x_1, \cdots, x_m : P_\alpha(x)]$ the group generated by $x_1, \cdots, x_m$ with $\{P_\alpha(x_1, \cdots, x_m) = 1\}$ as the set of defining relations. Since A is finitely generated, and h_1, h_2 are surjective, we can write

$$\begin{aligned} A &= [a_1, \cdots, a_m; P_\alpha(a)], \qquad B_1 = [b_1, \cdots, b_m; P_\alpha(b), Q_\beta(b)], \\ &\qquad B_2 = [c_1, \cdots, c_m; P_\alpha(c), R_\gamma(c)], \end{aligned} \tag{4.2}$$

where $b_i = h_1(a_i)$, $c_i = h_1(a_i)$, and $\{P_\alpha\}$, $\{Q_\beta\}$, $\{R_\gamma\}$ are three sets of words in m variables. It is evident that

$$B_1 \square B_2 = [b_1, \cdots, b_m, c_1, \cdots, c_m; P_\alpha(b), P_\alpha(c), Q_\beta(b), R_\gamma(c)]$$

and that N is the least normal subgroup of $B_1 \square B_2$ containing $b_1c_1^{-1}$, $\cdots, b_mc_m^{-1}$. Therefore

$$(B_1 \square B_2)/N \approx [b_1, \cdots, b_m, c_1, \cdots, c_m;$$
$$P_\alpha(b), P_\alpha(c), Q_\beta(b), R_\gamma(c), b_1 c_1^{-1}, \cdots, b_m c_m^{-1}]$$
$$\approx [b_1, \cdots, b_m; P_\alpha(b), Q_\beta(b), R_\gamma(b)]$$
$$\approx [a_1, \cdots, a_m; P_\alpha(a), Q_\beta(a), R_\gamma(a)].$$

On the other hand, we see immediately from (4.2) that K_1 and K_2 are, respectively, the least normal subgroups of A containing the sets $\{Q_\beta(a_1, \cdots, a_m)\}$ and $\{R_\gamma(a_1, \cdots, a_m)\}$ of elements. It follows then that $(B_1 \square B_2)/N \approx AK_1K_2$ which proves our proposition.

Now we return to the discussion of the transformation group of spheres, and let $G, S^n, X, X_1, X_2, \cdots$ have the same meaning as before.

(4.3) *Let $f_i^{\#}: \pi_1(X) \to \pi_1(X_i)$ be the homomorphisms of the fundamental groups induced by the projection $f_i: X \to X_i$, and K_i be the kernel of $f_i^{\#}$ $(i = 1, 2)$. Then $\pi_1(X) = K_1 \cdot K_2$, and $f_1^{\#}(K_2) = \pi_1(X_1)$, $f_2^{\#}(K_1) = \pi_1(X_2)$.*

Proof. We recall that $S^n = C_1 \cup C_2$, $C_1 \cap C_2 = X$. Let $\pi_1(C_1) \square \pi_1(C_2)$ denote the free product of $\pi_1(C_1)$ and $\pi_2(C_2)$, and N the least normal subgroup of $\pi_1(C_1) \square \pi_1(C_2)$ containing all the elements of the form $k_1^{\#}(a) k_2^{\#}(a^{-1})$, $a \in \pi_1(X)$, where $k_i^{\#}: \pi_1(X) \to \pi_1(C_i)$ are the homomorphisms of the fundamental groups induced by the injection map $k_i: X \to C_i$ $(i = 1, 2)$. Since $C_i - X$ is open in S^n and X, C_i are locally contractible, a theorem of van Kampen [12] tells us that $\pi_1(S^n) = \pi_1(C_1 \cup C_2) \approx (\pi_1(C_1) \square \pi_1(C_2))/N$. Hence

$$N = \pi_1(C_1) \square \pi_1(C_2). \tag{4.4}$$

Let $r_i: C \to X_i$ be the retraction of the mapping cylinder onto its upper base. We know that r_i is homotopic to the identity map of C_i, whence $r_i^{\#}: \pi_1(C_i) \to \pi_1(X_i)$ is bijective $(i = 1, 2)$. Since $f_i = r_i k_i$, the equality (4.4) tells us that

$$N^* = \pi_1(X_1) \square \pi_1(X_2), \tag{4.5}$$

where N^* is the least normal subgroup of $\pi_1(X_i) \square \pi_1(X_2)$ containing all the elements of the form $f_1^{\#}(a) f_2^{\#}(a^{-1})$, $a \in \pi_1(X)$. We know that the projection $f_i: X \to X_i$ is a fibre map with the homotopy sphere of positive dimension as the fibre, and therefore $f_i^{\#}$ is surjective. It follows from (4.1) and (4.5) that $\pi_1(X) = K_1K_2$. Hence

$$\pi_1(X_1) = f_1^{\#}(\pi_1(X)) = f_i(K_1K_2) = f_1^{\#}(K_2), \pi_1(X_2) = f_2^{\#}(\pi_1(X)) = f_2^{\#}(K_1K_2)$$
$$= f_2^{\#}(K_i).$$

(4.6) *Both $\pi_1(X_1)$ and $\pi_1(X_2)$ are cyclic, and $K_1 \cap K_2$ is the commutator subgroup of $\pi_1(X)$. Let $p = \dim X - \dim X_1$, $q = \dim X - \dim X_2$. Then* (i) *for $p > 1$ and $q > 1$, the orbits X, X_1, X_2 have trivial fundamental group, and* (ii) *when $p = 1$ and X_1 orientable, the circle bundle X over X_1 with the projection f_1 is the product bundle and $\pi_1(X)$ is abelian.*

Proof. We know that X is a fibre bundle over X_1 with the homotopy sphere S^p as the fibre, and thus we have the following exact sequence

$$\cdots \to \pi_1(S^p) \to \pi_1(X) \xrightarrow{f_1^{\#}} \pi_1(X_1) \to \cdots.$$

The kernel K_1 of $f_1^{\#}$, being a homomorphic image of $\pi_1(S^p)$, is cyclic. It follows then from (4.3) that $\pi_1(X_2)$ is cyclic. Similarly, we can show that $\pi_1(X_1)$ is cyclic. To see the intersection $K_1 \cap K_2$, let us denote by $f_{i*}: H_1(X, Z) \to H_1(X_i, Z)$ the homomorphisms of the 1-dimensional homology groups induced by $f_i: X \to X_i$. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{f^{\#}} & \pi_1(X_1) \times \pi_1(X_2) \\ \Big\downarrow m & & \Big\downarrow l \\ H_1(X, Z) & \xrightarrow{f_*} & H_1(X_1, Z) + H_1(X_2, Z), \end{array}$$

where m, l are the natural homomorphisms of the fundamental groups onto the 1-dimensional integral homology group, and $f^{\#}$, f_* are defined by

$$f^{\#}(a) = (f_1^{\#}(a), f_2^{\#}(a)), a \in \pi_1(X); \qquad f_*(f_{1*}(x), f_{2*}(x)), x \in H_1(X, Z);$$

the mapping $f_{i*}: H_1(X, Z) \to H_1(X_i, Z)$ denoting the homomorphisms induced by f_i. Since $\pi_1(X_1)$, $\pi_1(X_2)$ are abelian, l is bijective, and by (3.4), f_* is also bijective. Therefore, $f^{\#}$ and m have the same kernel. But the kernel of m is the commutator subgroup of $\pi_1(X)$ while the kernel of $f^{\#}$ is the intersection $K_1 \cap K_2$. The first part of our proposition is thus proved.

If $p > 1$, $q > 1$, it follows directly from the homotopy sequences for the bundles $\{X, X_i, f_i\}$ $(i = 1, 2)$ that $K_1 = K_2 = 0$, and hence by (4.3) that

$$\pi_1(X) = \pi_1(X_1) = \pi_1(X_2) = 0.$$

Now suppose that $p = 1$ and X_1 is orientable. Then X is an orientable circle bundle over X, hence a principal bundle. From (3.5), this bundle has zero characteristic class, and then it must be the product bundle. It follows that $\pi_1(X)$ is isomorphic with the direct product of $\pi_1(X_1)$ and a free cyclic group and therefore is abelian.

(4.7) *Suppose that X_1 is not a point. Then the map $f: X \to X_1 \times X_2$ defined by $f(x) = (f_1(x), f_2(x))$, $x \in X$, is a covering map of X onto $f(X)$.*

Proof. Let us write X, X_1, X_2 in the form of coset spaces G/L, G/L_1, G/L_2. Then we see immediately that $f(gL) = f(g'L)$ if and only if $g^{-1}g' \in L_1 \cap L_2$, where $g, g' \in G$. Thus $f: X \to f(X)$ is a fibre map with $(L_1 \cap L_2)/L$ as the fibre. To prove our proposition (4.7), it suffices to prove that $(L_1 \cap L_2)/L$ is finite. We know that $f_1: G/L \to G/L_1$ is a fibre map with the homotopy sphere S^p as the fibre. Choose the particular fibre $F_1 = \{gL: g \in L_1\}$ in G/L. On account of (3.5) and (3.6), the fundamental cycle of F_1 gives a non-zero element y in the p-dimensional homology group $H_p(G/L, A)$ with coefficient group A, where A denotes the free cyclic group or the cyclic group of order two according as G/L_1 is orientable or not. Let $f_{i*}: H_p(G/L, A) \to H_p(G/L_i, A)$ be the homomorphism of the homology groups induced by the projection $f_i: G/L \to G/L_i$ $(i = 1, 2)$, and

$$f_*: H_p(G/L, A) \to H_p(G/L_1, A) + H_p(G/L_2, A)$$

be defined by $f_*(u) = (f_{1*}(u), f_{2*}(u))$, $u \in H_p(G/L, A)$. By our assumption, X_1 is not a point so $p < n-1$, and then from the dual of (3.4), f_* is injective. Since $f_{1*}(y) = 0$, $f_{2*}(y)$ cannot be zero and therefore $f_2(F_1)$ must have dimension $\geqq p$. On the other hand, we see directly that $f_2(F_1)$ is homeomorphic with the coset space $L_1/(L_1 \cap L_2)$. This tells us that

$$\dim(L_1 \cap L_2) \leqq \dim L_1 - p.$$

But $\dim L = \dim L_1 - p$ and $L \subset L_1 \cap L_2$, whence $\dim(L_1 \cap L_2) = \dim L$. It follows that $(L_1 \cap L_2)/L$ is finite and thus (4.7) is proved.

(4.8) *If $\dim X_1 + \dim X_2 = \dim X$, then f is a homeomorphism of X onto $X_1 \times X_2$. Therefore, both bundles $\{X, X_1, f_1\}$, $\{X, X_2, f_2\}$ are product bundles and X_1, X_2 are spheres.*

Proof. Since $\dim X_2 < \dim X$, X_1 cannot be a point, and therefore $f(X)$ has the same dimension as X. From our hypothesis, $f(X)$ is a closed submanifold of $X_1 \times X_2$ of the same dimension, whence $f(X) = X_1 \times X_2$. Thus X is a covering space of $X_1 \times X_2$. Identifying $\pi_1(X_1 \times X_2)$ with $\pi_1(X_1) \times \pi_1(X_2)$, we know that the homomorphism $f^{\#}: \pi_1(X) \to \pi_1(X_1 \times X_2)$ induced by the map $f: X \to X_1 \times X_2$ is given by $f^{\#}(a) = (f_1{}^{\#}(a), f_2{}^{\#}(a))$, $a \in \pi_1(X)$. By (4.3) $f^{\#}$ is surjective, and therefore the covering map $f: X \to X_1 \times X_2$ must be a homeomorphism.

Let $l_i: X_1 \times X_2 \to X_i$ be the projection $(i = 1, 2)$. Then f establishes an

equivalence of the bundle $\{X, X_i, f_i\}$ with the product bundle $\{X_1 \times X_2, X_i, l_i\}$. (4.8) is thus proved.

(4.9) *Let L^0, L_i^0 denote, respectively, the identity components of L, L_i, and $\Pi_i = (L_i^0 \cap L)/L^0$. Then $(L/L^0)/\Pi_i \approx L_i/L_i^0$ is cyclic $(i = 1, 2)$ and $L/L^0 = \Pi_1 \cdot \Pi_2$.*

Proof. Let us consider the following commutative diagram

$$\begin{array}{ccccc}
 & & & \xrightarrow{\theta_1} & \\
\pi_1(G) & \longrightarrow & \pi_1(X_1) & \longrightarrow & L_1/L_1^0 \\
 & & \uparrow{\scriptstyle f_1^{\#}} & & \uparrow{\scriptstyle \epsilon_1} \\
\pi_1(G) & \longrightarrow & \pi_1(X) & \xrightarrow{\theta} & L/L^0 \\
 & & \downarrow{\scriptstyle f_2^{\#}} & & \downarrow{\scriptstyle \epsilon_2} \\
 & & & \xrightarrow{\theta_2} & \\
\pi_1(G) & \longrightarrow & \pi_1(X_2) & \longrightarrow & L_2/L_2^0,
\end{array}$$

where the horizontal sequences are the homotopy sequences associated to the bundles $X_1 = G/L_1$, $X = G/L$, $X_2 = G/L_2$, and ϵ_1, ϵ_2 are the natural projections. Evidently Π_i is the kernel of ϵ_i, and since θ, θ_1, θ_2 are surjective, we have $(L/L^0)/\Pi_i \approx \epsilon_i(L/L^0) = L_i/L_i^0$. By (4.6), $\pi_1(X_i)$ is cyclic, so L_i/L_i^0 is also cyclic. To see the last part of (4.9), let K_i denote the kernel of $f_i^{\#}$. Then $\theta(K_i) \subset \Pi_i$ and then from (4.3) it follows that

$$\Pi_1 \cdot \Pi_2 \supset \theta(K_1) \cdot \theta(K_2) = \theta(\pi_1(X)) = L/L^0,$$

which completes the proof of (4.9).

5. The orbits under G when they are all simply connected. In this section and the next, we shall use the known results on the real cohomology algebras of the homgeneous spaces of compact Lie groups established by Samelson, Leray, Koszul and H. Cartan. The notations of H. Cartan [4] will be used throughout. In fact let Q be a compact Lie group. We shall always use $S(Q)$ to denote the algebra of real symmetric multilinear forms over the Lie algebra of Q, and $I(Q)$ to denote the subalgebra of $S(Q)$ consisting of the fixed elements under the adjoint group of Q. For any principal bundle with base B and group Q, there is a characteristic map $\sigma: I(Q) \to H^*(B)$, where H^* denotes the real cohomology algebra. The image of σ is called the *characteristic subalgebra* of $H^*(B)$.

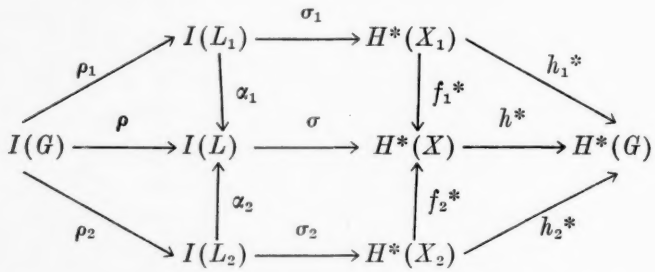
Now we consider our transformation group G of S^n with $(n-1)$-dimensional orbits. Let $X, X_1, X_2, L, L_1, L_2, \cdots$ retain their meaning as

before, and let

$$p = \dim X - \dim X_1 = \dim L_1 - \dim L,$$

$$q = \dim X - \dim X_2 = \dim L_2 - \dim L.$$

Assume $p > 1$, $q > 1$. It follows from (4.6) that X, X_1, X_2 are simply connected, and hence L, L_1, L_2 are connected. From [4], we have the following commutative diagram



where H^* denotes the real cohomology algebra, ρ, ρ_1, ρ_2, α_1, α_2 the restriction maps, σ, σ_1, σ_2 the characteristic maps, and h^*, h_1^*, h_2^* the homomorphisms induced, respectively, by the natural projections $h: G \to X$, $h_1: G \to X_1$, $h_2: G \to X_2$.

Suppose that both p and q are odd. L_1/L and L_2/L are odd-dimensional spheres and therefore $\alpha_1: I(L_1) \to I(L)$ and $\alpha_2: I(L_2) \to I(L)$ are surjective. Let K, K_1, K_2 be, respectively, the characteristic subalgebras of $H^*(X)$, $H^*(X_1)$, $H^*(X_2)$. Then we have, from diagram (5.1),

$$f_1^*(K_1) = f_1^*\sigma_1(I(L_1)) = \sigma\alpha_1(I(L_1)) = \sigma(I(L))$$

$$= K = \sigma\alpha_2(I(L_2)) = f_2^*\sigma_2(I(L_2)) = f_2^*(K_2).$$

On account of (3.4), K, K_1, K_2 can contain only elements of zero degree, and hence h^*, h_1^*, h_2^* are injective. Let $P(G)$, $P_1(G)$, $P_2(G)$ denote, respectively, the linear subspace of $h^*(H^*(X))$, $h_1^*(X^*(X_1))$, $h_2^*(H^*(X_2))$ consisting of the primitive elements of $H^*(G)$. By a theorem of Samelson [13], we have for $i = 1, 2$,

$$H^*(X) \approx h^*(H^*(X)) \approx \Lambda(P(G)),$$

$$H^*(X_i) \approx h_i^*(H^*(X_i)) \approx \Lambda(P_i(G)),$$

and hence

$$\dim H^*(X) = 2^b, \quad \dim H^*(X_i) = 2^{b_i}, \quad b = \dim P(G), \quad b_i = \dim P_i(G),$$

where Λ denotes the exterior algebra, and dimension here is used in the sense

of that of linear spaces over the real field. Evidently, $P_1(G) + P_2(G) \subset P(G)$, and from (3.4), $P_1(G) \cap P_2(G) = 0$. Therefore, $\dim H^*(X) \geqq 2^{b_1+b_2}$. On the other hand, we know from (3.5) that $\dim H^*(X) = 2 \dim H^*(X_i)$, whence $b_1 \leqq 1$, $b_2 \leqq 1$.

If one of b_1, b_2 is equal to zero, say b_i, then X_1 is reduced to a point on account of its orientability, and so is X_2. Now assume that none of b_1, b_2 is equal to zero. Then $b_1 = b_2 = 1$, or what is the same, there exist positive integers r, s such that $H^*(X_1) \approx H^*(S^r)$, $H^*(X_2) \approx H^*(S^s)$. It follows then from (3.4) that the Poincaré polynomial of X is $1 + t^r + t^s + t^{n-1}$. But X, X_1, X_2 are all orientable, and so $r + s = n - 1$, $r = \dim X_1$, $s = \dim X_2$. Proposition (4.8) then tells us that X_1 is a q-sphere, X_2 is a p-sphere and X is homeomorphic with the product $X_1 \times X_2$.

Thus we have proved

(5.2) *Let $p = \dim X - \dim X_1$, $q = \dim X - \dim X_2$. If $p > 1$, $q > 1$, and both p and q are odd, then either* (i) *X_1, X_2 are points, or* (ii) *X_1 is a q-sphere, X_2 is a p-sphere and X is homeomorphic with the product $X_1 \times X_2$.*

6. The case by which both p and q are even.

(6.1) *Suppose that both p and q are even. Then X, X_1, X_2 have non-vanishing Euler characteristic.*

Proof. Let $\mathfrak{P}(t)$, $\mathfrak{P}_1(t)$, $\mathfrak{P}_2(t)$ be the Poincaré polynomials of X, X_1, X_2 respecetively. From (3.4) and (3.5), we have

$$\mathfrak{P}(t) = \mathfrak{P}_1(t) + \mathfrak{P}_2(t) + t^{n-1}, \qquad \mathfrak{P}(t) = (1 + t^p)\mathfrak{P}_1(t) = (1 + t^q)\mathfrak{P}_2(t).$$

Let $\mathfrak{Q}(t)$ be the polynomial obtained from $\mathfrak{P}(t)$ by omitting all the terms of even degree, and $\mathfrak{Q}_1(t)$, $\mathfrak{Q}_2(t)$ are similarly obtained from $\mathfrak{P}_1(t)$, $\mathfrak{P}_2(t)$. since p and q are even,

$$\mathfrak{Q}(t) = (1 + t^p)\mathfrak{Q}_1(t) = (1 + t^q)\mathfrak{Q}_2(t),$$

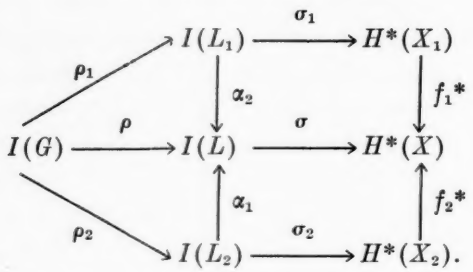
$$\mathfrak{Q}(t) = \begin{cases} \mathfrak{Q}_1(t) + \mathfrak{Q}_2(t), & \text{if } n-1 \text{ is even} \\ \mathfrak{Q}_1(t) + \mathfrak{Q}_2(t) + t^{n-1}, & \text{if } n-1 \text{ is odd.} \end{cases}$$

Suppose that $n - 1$ is even. Then we have $t^{p+q}\mathfrak{Q}_1(t) = \mathfrak{Q}_1(t)$ which implies that $\mathfrak{Q}(t) = \mathfrak{Q}_1(t) = \mathfrak{Q}_2(t) = 0$. Therefore X, X_1, X_2 have non-vanishing Euler characteristic if $n - 1$ is even.

Now suppose that $n - 1$ is odd. A direct calculation tells us that $(t^{p+q} - 1)\mathfrak{Q}_1(t) = t^{n-1}(1 + t^q)$. This is impossible as $t^{n-1}(1 + t^q)$ is not divisible by $t^{p+q} - 1$. Therefore $n - 1$ cannot be odd, and the proof is thus completed.

(6.2) *Suppose that p and q are even. Then L_1 and L_2 generate the entire group G.*

Proof. Let us consider the commutative diagram



On account of (3.4), we have

$$H^0(X) \supset f_1^*\sigma_1(I(L_1)) \cap f_2^*\sigma_2(I(L_2))$$
$$= \sigma\alpha_1(I(L_1)) \cap \sigma\alpha_2(I(L_2)) = \sigma(\alpha_1(I(L_1)) \cap \alpha_2(I(L_2))).$$

Denote by $I^+(G)$ the ideal of $I(G)$ consisting of elements of positive degree, and by J the ideal of $I(L)$ generated by $\rho(I^+(G))$. Then [4] J is the kernel of σ, and hence

$$\alpha_1(I^+(L_1)) \cap \alpha_2(I^+(L_2)) \subset J.$$

Since X, X_1, X_2 have non-vanishing Euler characteristic, the groups G, L, L_1, L_2 have the same rank. It follows that the maps ρ, ρ_1, ρ_2, α_1, α_2 are injective and the characteristic maps σ, σ_1, σ_2 are surjective. Thus we can consider $I(G)$, $I(L_1)$, $I(L_2)$ simply to be subalgebras of $I(L)$, and neglect the identity maps ρ, ρ_1, ρ_2, α_1, α_2. Then

$$I^+(G) \subset I^+(L_1) \cap I^+(L_2) \subset J = I(L) \cdot I^+(G),$$

and then

$$J = I(L) \cdot I^+(G) \subset I(L) \cdot \{I^+(L_1) \cap I^+(L_2)\} \subset J,$$

or what is the same,

$$I(L) \cdot \{I^+(L_1) \cap I^+(L_2)\} = J = \text{kernel of } \sigma. \tag{6.3}$$

Let Q be the least closed subgroup of G which contains L_1, L_2. Since both L_1 and L_2 are connected, Q is connected. Then $I(G) \subset I(Q) \subset I(L_1) \cap I(L_2)$, whence $J = I(L) \cdot I^+(Q)$. L and Q have the same rank so

$$H^*(X) = \sigma(I(L)) \approx I(L)/J = I(L)/(I(L) \cdot I^+(Q)) \approx H^*(Q/L).$$

Therefore, $\dim X = \dim Q/L$, $\dim G = \dim Q$ and $G = Q$. (6.2) is thus proved.

For further discussions of the orbits, we need the following:

(6.4) *Let U be a compact, connected Lie group, and Q a closed normal subgroup of U. There exists a connected, closed, normal subgroup M of U such that*

$$U = M \cdot Q, \qquad M \cap Q = \text{finite}.$$

This M is unique unless both Q and U/Q have center of positive dimension.

Proof. This can be proved directly by using Lie algebras.

7. Determination of the orbits when p and q are even. Now we are in a position to determine the orbits X, X_1, X_2 when p and q are even. We assume that G acts effectively on S^n and so it acts effectively on X. Let N_1 be the largest normal subgroup of L_1 which is contained in L. This subgroup N_1 can also be regarded as the maximal subgroup of L_1 acting trivially on L_1/L. Thus L_1/N_1 is an effective and transitive transformation of the p-sphere L_1/L with L/N_1 as the isotropic subgroup. Suppose that N_2 is similarly defined.

(7.1) *If p is even and $\neq 6$, then $(L_1/N_1, L/N_1) = (O(p+1), O(p))$. If $p = 6$, then $(L_1/N_1, L/N_1)$ is either $(O(7), O(6))$ or (G_2, A_2), where $O(m)$ denotes the proper orthogonal group of degree m, G_2 the exceptional Lie group of dimension 16, and A_2 the unimodular unitary group of rank 2.*

Proof. This is a known result [16].

(7.2) *Suppose that p and q are even and G acts effectively on S^n. Then $N_1 \cap N_2 = e$, and either* (i) $L = N_1 \cdot N_2$, *or* (ii) $p = q$, *N_1 and N_2 are finite, or* (iii) $p = q = 4$, *N_1 and N_2 are locally isomorphic with $O(3)$, and L is locally isomorphic with $O(3) \times O(3) \times O(3)$.*

Proof. Since L_1/N_1 has no center, by (6.4), there exists a unique connected normal subgroup M_1 of L_1 such that $L_1 = M_1 N_1$ and the intersection $M_1 \cap N_1$ is a finite central subgroup of L_1. We know that N_1, N_2 are normal subgroups of L and so $N_1 \cap N_2$ is normal in N_1. Since M_1 and N_1 commute elementwise, it follows that $N_1 \cap N_2$ is a normal subgroup of L_1. Similarly, we can prove that $N_1 \cap N_2$ is a normal subgroup of L_2. On account of (6.2), $N_1 \cap N_2$ must be normal in G. But $N_1 \cap N_2 \subset L$ and G acts effectively on X. It follows that $N_1 \cap N_2 = e$.

To prove the others parts of our proposition, let us assume that $L \neq N_1 N_2$. We find it convenient to divide it into two cases and discuss them separately.

Case 1. Either N_1 or N_2 is finite. To be precise, let N_1 be finite. Then L_1 acts almost effectively on the p-sphere L_1/L. Hence L is simple when $p \neq 4$ and is locally isomorphic with $O(4)$ when $p = 4$. In the former alternative, the subgroup N_2, being normal in L, can only be finite or the whole group L. But by our assumption, $L \neq N_2$ and so N_2 must be finite. In the second alternative, N_2 can only be finite or locally isomorphic with $O(3)$. If N_2 is locally isomorphic with $O(3)$, then L/N_2 is locally isomorphic with $O(4)/O(3)$ which contradicts the fact that $(L_2/N_2)/(L/N_2) = L_2/L$ is an even sphere. Hence both N_1 and N_2 are finite in any alternative. It follows that L/N_1 and L/N_2 are locally isomorphic, and then $p = q$ on account of (7.1).

Case 2. Both N_1 and N_2 are infinite. Consider the projection ϕ: $L \to L/N_1$. Then $\phi(N_2)$ is a normal subgroup of L/N_1. Since $N_1 \cap N_2 = e$, $\phi(N_2)$ is isomorphic with N_2, and hence infinite. On the other hand, from our assumption $L \neq N_1 N_2$, we know that $\phi(N_2) \neq L/N_1$. It follows that L/N_1 cannot be simple, and then $p = 4$ and $N_2 \approx \phi(N_2)$ is locally isomorphic with $O(3)$. Similarly, we can prove that $q = 4$, and N_1 is locally isomorphic with $O(3)$. By (7.1), $L/N_1 \approx O(4)$ and so L is locally isomorphic with $O(3) \times O(3) \times O(3)$. This completes the proof of (7.2).

(7.3) *Suppose that p, q are even and $L = N_1 \times N_2$. Then either* (i) *G is of rank 2 and L is a two-dimensional torus, or* (ii) *X_1 is a q-sphere and X_2 a p-sphere.*

Proof. From (6.4), there exists a connected normal subgroup M of L_1 such that $L_1 = N_1 M$ and $N_1 \cap M$ is a finite central subgroup of L_1. Let V be the identity component of $L \cap M$. Since $L = N_1 \times N_2 \subset L_1$, we see immediately that

$$L = N_1 \cdot V, \qquad N_1 \cap V = \text{finite}.$$

Taking account of the equality $L = N_1 \times N_2$, we know from (6.4) that either $V = N_2$ or both N_1, N_2 have infinite center. Let us discuss these two cases separately.

Case 1. Both N_1 and N_2 have infinite center. We know that L_1/N_1 acts transitively and effectively on the p-sphere L_1/L with L/N_1 as isotropic subgroup. Since $L/N_1 \approx N_2$ has infinite center, p must be 2 and L/N_1 must be the circle group on account of (7.1). Similarly, $q = 2$ and L/N_2 is

the circle group. Hence N_1, N_2 are circle groups, L is the two-dimensional toral group and G of rank 2.

Case 2. Either N_1 or N_2 has a finite center. Then from the above discussions, we know that $N_2 = V$, or what is the same $N_2 \subset M$. Since L and L_1 have the same rank, M and N_2 have the same rank. Therefore N_2 contains the center of M, and hence contains $M \cap N_1$. Then $M \cap N_1 = M \cap N_1 \cap N_2 = e$ which tells us that $L_1 = N_1 \times M$. Similarly, there exists a connected normal subgroup M' of L_2 such that $L_2 = M' \times N_2$ and $N_1 \subset M'$. It is to be noted that we do not know whether M and M' commute.

Now let use use the tensor notation to write down the algebras of the invariant symmetric multilinear forms:

$$I(L) = I(N_1) \otimes I(N_2),\ I(L_1) = I(N_1) \otimes I(M),\ I(L_2) = I(M') \otimes I(N_2).$$

Thus we have the following commutative diagram

$$\begin{array}{ccc}
I(L_1) = I(N_1) \otimes I(M) & \xrightarrow{\sigma_1} & H^*(X_1) \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle f_1^*} \\
I(L) = I(N_1) \otimes I(N_2) & \xrightarrow{\sigma} & H^*(X) \\
\uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle f_2^*} \\
I(L_2) = I(M') \otimes I(N_2) & \xrightarrow{\sigma_2} & H^*(X_2).
\end{array}$$

Since

$$N_2 \subset M,\ N_1 \subset M',\ \text{rank}(M) = \text{rank}(N_2),\ \text{rank}(M') = \text{rank}(N_1),$$

we can consider $I(M)$ to be a subalgebra of $I(N_2)$, and $I(M')$ to be a subalgebra of $I(N_1)$. Therefore,

$$I(L_1) \cap I(L_2) = I(M') \otimes I(M) \subset I(N_1) \otimes I(N_2).$$

Let $J_1 = I(N_1) \cdot I^+(M')$, $J_2 = I(N_2) \cdot I^+(M)$ be, respectively, the ideals generated by $I^+(M')$, $I^+(M)$ in $I(N_1)$, $I(N_2)$. Then

$$I(L) \cdot \{I^+(L_1) \cap I^+(L_2)\} = J_1 \otimes I(N_2) + I(N_1) \otimes J_2.$$

Taking account of (6.3), we obtain then

$$\begin{aligned} H^*(X) \approx I(L)/J &= \{I(N_1) \otimes I(N_2)\}/\{J_1 \otimes I(N_2) + I(N_1) \otimes J_2\} \\ &\approx I(N_1)/J_1 \otimes I(N_2)/J_2. \end{aligned}$$

On the other hand, M/N_2 is homeomorphic with the p-sphere L_1/L, and M'/N_2

with the q-sphere L_2/L. Since both p and q are even, $H^*(M/N_2)$ and $H^*(M'/N_1)$ coincide with their respective characteristic subalgebras, i.e.,

$$H^*(S^p) \approx H^*(M/N_2) \approx I(N_2)/J_2, \qquad H^*(S^q) \approx H^*(M'/N_1) \approx I(N_1)/J_1.$$

It follows then that $H^*(X) \approx H^*(S^p) \otimes H^*(S^q)$, and then by (3.5), $H^*(X_1) \approx H^*(S^q)$, $H^*(X_2) \approx H^*(S^p)$. This tells us that $\dim X_1 + \dim X_2 = \dim X$. From (4.8), we know then that $X_1 = S^q$, $X_2 = S^p$, $X = S^p \times S^q$.

(7.4) *Suppose that p and q are even and G acts effectively on S^n. Let us use $A_m(B_m, C_m, D_m, G_2, F_4, E_6, E_7, E_8)$ to denote a compact group in the Cartan's class $A_m(B_m, C_m, D_m, G_2, F_4, E_6, E_7, E_8)$, where the lower index signifies the rank. Then there are only five possible cases, namely:*

(I) *X_1 is a q-sphere, X_2 is a p-sphere and $X = X_1 \times X_2$;*

(II) *X_1 and X_2 are single points and X is an $(n-1)$-sphere;*

(III) *G is a compact simple Lie group of rank 2 without center, and L is the two-dimensional toral group, and L_1, L_2 are locally isomorphic with the product of a circle group and A_1;*

(IV) *$p = q = 4$, $n = 13$, G is the adjoint group of C_3, L is locally isomorphic with $C_1 \times C_1 \times C_1$, L_1 and L_2 are locally isomorphic with $C_1 \times C_2$, and the orbits X_1, X_2 are quaternion projective planes;*

(V) *$p = q = 8$, $n = 25$, $G = F_4$, $L = D_4$, $L_1 = B_4$, $L_2 = B_4$, and the orbits X_1, X_2 are Cayley projective planes.*

Proof. From (7.2), we know that there are only three possibilities: (i) $L = N_1 \times N_2$, (ii) $p = q$, N_1 and N_2 are finite, and (iii) $p = q = 4$, N_1 and N_2 are both C_1, and L is locally isomorphic with $C_1 \times C_1 \times C_1$. Let us consider them separately.

(i) In this possibility, we know from (7.3) that either $X_1 = S^q$, $X_2 = S^p$, $X = S^q \times S^p$, or G is of rank 2 and L is the two-dimensional toral group T^2. The first alternative is the case (I) in the above list, and therefore let us assume that G is of rank 2 and $L = T^2$. Since G acts effectively on X, and X has positive Euler characteristic, G has a trivial center, and in particular, is semi-simple. If G is simple, then it is the case (III). If G is not simple, then $G = D_2$, $X = D_2/T^2 = S^2 \times S^2$, whence $X_1 = X_2 = S^2$. This is included in the case (I) of the above list.

(ii) In this case, $p = q$, N_1 and N_2 are finite. Since N_1 is finite, L_1 acts almost effectively on the p-sphere L_1/L, and hence L_1 is either G_2 or a

B_m, where $m = p/2$. We know that $L_1 \subset G$ and G, L_1 have the same rank. If $L_1 = G_2$, then $G = L_1$ (because no other group of rank 2 can contain G_2 as a proper subgroup), whence X_1 is a single point, and we get case (II). Suppose $L_1 = B_m$. We know [3] that there is no compact connecter Lie group of rank m which can contain B_m as a proper subgroup unless $m = 4$, and when $m = 4$, F_4 is the only group having these properties. Therefore, either $G = L_1$, or $(G, L_1) = (F_4, B_4)$. In the first alternative, X_1 is reduced to a point, and we get case (II). In the second alternative, $G = F_4$, $L = B_4$, whence $p = q = 8$, and $L = D_4$, $L_2 = B_4$ on account of (7.1). The orbits X_1, X_2, being homeomorphic with F_4/B_4, are the Cayley projective planes. This is the case (V) in our list.

(iii) In this case, $p = q = 4$, $N_1 = C_1$, $N_2 = C_1$ and L is locally isomorphic with $C_1 \times C_1 \times C_1$. Since L_1/N_1 acts transitively and effectively on the 4-sphere L_1/L, L_1 is locally isomorphic with $C_1 \times C_2$. The only connected and compact Lie group which can contain L_1 as a proper subgroup of the same rank is C_3 [3]. Therefore, either $G = L_1$ or $G = C_3$. In the first alternative, X_1 is a point, and we get case (II). In the second alternative, $G = C_3$, L_1 and L_2 are locally isomorphic with $C_1 \times C_2$. The exceptional orbits X_1 and X_2, being homeomorphic with $C_3/C_1 \times C_2$, are quaternion projective planes. This is the case (IV) in the above list.

(7.5) *In the Case* III *of* (7.4), *the connected center of at least one of* L_1, L_2 *is not contained in the semi-simple part of the other.*

Proof. Let us write $L_1 = N_1 M$, $L_2 = M' N_2$, where M, M' are, respectively, the largest connected normal subgroups of L_1, L_2 which act almost effectively on L_1/L, L_2/L. From the proof of (7.4), we know that $L = N_1 \times N_2 = T^2$. Since both L_1 and L_2 are locally isomorphic with $T \times A_1$, it follows that N_1 and N_2 (M and M') are, respectively, the connected center (semi-simple part) of L_1 and L_2. Suppose (7.5) to be false. Then $N_1 \subset M'$, $N_2 \subset M$. By using exactly the same reasoning as in the proof of Case 2 of (7.3), we know that X_1 and X_2 must be spheres. A contradiction is thus obtained, and (7.5) is proved.

8. The case by which q is even and $p > 1$ is odd. Suppose that p is odd and q is even. Then L_1/L is an odd sphere and L_2/L an even sphere. Theerfore, in the diagram (5.1), α_1 is surjective and α_2 injective and

$$f_2^*(K_2) = f_2^*\sigma_2(I(L_2)) = \sigma\alpha_2(I(L_2)) \subset \sigma(I(L))$$
$$= f_1^*\sigma_1(I(L_1)) = f_1^*(K_1),$$

where K_1, K_2 denote the characteristic subalgebras of the homogeneous spaces $X_1 = G/L_1$, $X_2 = G/L_2$ respectively. From (3.4), the common elements of the images of f_1^* and f_2^* are of zero degree. Therefore, K_2 as well as $f_2^*(K_2)$ contains only elements of zero degree. It follows that h_2^* is injective and L_2 is nonhomologous to zero in G.

Let $\mathfrak{P}(X)$, $\mathfrak{P}(X_1)$, $\mathfrak{P}(X_2)$ be the Poincaré polynomials of X, X_1, X_2 respectively with t as the indeterminate. Then from (3.4) and (3.5), we have

$$\mathfrak{P}(X) = (1+t^p)\mathfrak{P}(X_1) = (1+t^q)\mathfrak{P}(X_2), \tag{8.1}$$

$$\mathfrak{P}(X) = \mathfrak{P}(X_1) + \mathfrak{P}(X_2) + t^{n-1} - 1. \tag{8.2}$$

Let s be the largest integer such that $(1+t)^s$ divides $\mathfrak{P}(X_2)$. Since p is odd, $s \geqq 1$. From (8.1), $(1+t)^{s-1}$ divides $\mathfrak{P}(X)$, $\mathfrak{P}(X_1)$ and $\mathfrak{P}(X_2)$, and so $(1+t)^{s-1}$ divides $t^{n-1}-1$ on account of (8.2). It follows that s is equal to 1 or 2.

Now suppose that $s=1$. Since $L_2 \nsim 0$ in G, by Samelson's results, X_2 is a real homology sphere of odd dimension. We have then from (8.1) and (8.2) that $\mathfrak{P}(X_1) = 1+t^q$, $\mathfrak{P}(X_2) = 1+t^p$, $p+q = n-1$. Proposition (4.8) then tells us that $X_1 = S^q$, $X_2 = S^p$, $X = S^p \times S^q$.

Suppose that $s=2$. Then $\mathfrak{P}(X_2)$ takes the form $(1+t^u)(1+t^v)$, where u, v are odd. Taking account of (8.1) and (8.2), we can get by a direct computation that

$$\mathfrak{P}(X_1) = (1+t^q)(1+t^{p+q}),\ \mathfrak{P}(X_2) = (1+t^p)(1+t^{p+q}),\ 2(p+q) = n-1,$$
$$\mathfrak{P}(X) = (1+t^p)(1+t^q)(1+t^{p+q}).$$

Thus we have proved

(8.3) *Suppose that q is even, p is odd and greater than* 1. *Then either* (A) $X_1 = S^q$, $X_2 = S^p$, $X = S^p \times S^q$, *or* (B) L_2 *is non-homologous to zero in* G, $n = 2(p+q)+1$, *and*

$$\mathfrak{P}(X_1) = (1+t^q)(1+t^{p+q}), \qquad \mathfrak{P}(X_2) = (1+t^p)(1+t^{p+q}).$$

To study Case B in (8.3), we need to analyse the structures of G, L, L_1, L_2 in detail, and for this purpose, we find it convenient to lay down the following

DEFINITION. *Let W be a connected Lie group and $W_1, W_2, \cdots, W_r$ connected subgroups. We say that W is locally the direct product of $W_1, W_2, \cdots, W_r$ if the Lie algebra of W is the direct sum of the Lie algebras*

of the W_i's. If this is the case, we shall write $W = W_1 \cdot W_2 \cdot \cdots \cdot W_r$. We note that each W_i is a normal subgroup of W.

Now let us consider the subgroups L, L_1, L_2. Since both p and q are greater than one, they are connected. Let N_1 be the maximum connected normal subgroup of L_1 which acts trivially on L_1/L. Then we can write $L_1 = M \cdot N_1$, where M acts almost effectively on L_1/L. But L_1/L is an odd sphere, and so by Montgomery-Samelson theorem, we have $M = M_s \cdot J$, where M_s is simple and transitive on L_1/L while J is either the identity or a group of rank 1. Let $M' = M \cap L$, $M_s' = M_s \cap L$. Then $M' = M_s' \cdot J'$, $L = M' \cdot N_1$, where J' is a normal subgroup isomorphic with J. It is is to be noted that if $M_s' = e$, then $J = J' = e$ and L_1/L is a three-sphere. Similarly, we can write $L_2 = Q \cdot N_2$, $L = Q' \cdot N_2$, where Q acts almost effectively and transitively on the even sphere L_2/L while N_2 acts trivially, and $Q' = Q \cap L$.

(8.4) *The subgroups L, L_1, L_2 have local direct decomposition as follows:*

$$L_2 = Q \cdot N_2, \quad L_1 = M_s \cdot J \cdot N_1, \quad L = Q' \cdot N_2 = M_s' \cdot J' \cdot N_1,$$

where $J' \subset M_s \cdot J$, $Q/Q' = S^q$, $M_s/M_s' = S^p$ and $J \approx J'$ is of rank $\leqq 1$. If, moreover, $p + q > 15$, then M_s' is a normal subgroup of N_2 and Q' is a normal subgroup of $J' \cdot N_1$.

Proof. The first part is merely a summary of what we have just obtained. It remains only to show the second part. If M_s' is the identity, then this is evident. Otherwise, M_s' is a connected simple non-abelian normal subgroup of L. Therefore, M_s' is either a normal subgroup of Q' or that of N_2.[6] But the compact transitive transformation groups of spheres as well as their isotropic subgroups are completely known [1]. A direct verification tells us that when $p + q > 15$, M_s' and Q' cannot have any common normal subgroup of positive dimension. It follows that $M_s' \subset N_2$ and $Q' \subset J' \cdot N_1$.

(8.5) *Suppose that G acts almost effectively on S^n with an $(n-1)$-dimensional orbit, and G is a local direct product $G = G' \cdot G''$, where $G' \neq e$, $G'' \neq e$. If $L_i = L_i' L_i''$, $L = L'L''$, where L', $L_i' \subset G'$ and L'', $L_i'' \subset G''$ $(i = 1, 2)$, then $X_1 = S^q$, $X_2 = S^p$.*

Proof. The direct product $G' \times G''$ is a finite covering of G, and thus acts on S^n almost effectively with the same orbits as G. Moreover, the isotropic subgroups of $G' \times G''$ satisfy the conditions of our proposition (8.5). Therefore, we can simply take G to be the direct product $G' \times G''$. It is evident that

[6] This follows from the fact that $Q' = O(q)$ and $q > 2$.

$L' \subset L_i'$, $L'' \subset L_i''$, $X = (G'/L') \times (G''/L'')$, $X_i = (G'/L_i') \times (G''/L_i'')$, $(i = 1, 2)$. Let

$$j_i' : H^*(G'/L_i') \to H^*(G'/L'), \qquad j_i'' : H^*(G''/L_i'') \to H^*(G''/L_i'')$$

be the homomorphisms of the real cohomology induced by the inclusions of the isotropic subgroups. We have the following commutative diagram:

$$\begin{array}{ccc} H^*(X_1) & = & H^*(G'/L_1') \otimes H^*(G''/L_1'') \\ {\scriptstyle f_1^*}\downarrow & & \downarrow{\scriptstyle j_1' \otimes j_1''} \\ H^*(X) & = & H^*(G'/L') \otimes H^*(G''/L'') \\ {\scriptstyle f_2^*}\uparrow & & \uparrow{\scriptstyle j_2' \otimes j_2''} \\ H^*(X_2) & = & H^*(G'/L_2') \otimes H^*(G''/L_2''). \end{array}$$

Since f_1^*, f_2^* are injective, j_1', j_2', j_1'', j_2'' are all injective. The dimension of X being greater than the dimension of X_1, we can therefore, up to a change of notation, assume that $m = \dim G'/L' > \dim G'/L_1'$. Since X is orientable, there exists a non-zero element $u \in H^m(G'/L')$ of degree m. By our assumptions, $G'' \neq e$ and G acts almost effectively on S^n. Therefore G''/L'' cannot be a point and then $m < \dim X = n - 1$. It follows then from (3.4) that there exists $u_2 \in H^m(G'/L_2')$ such that $j_2'(u_2) = u$, and hence $\dim G'/L_2' \geqq m = \dim G'/L'$. This tells us that $\dim L_2' \leqq \dim L'$, whence $L_2' = L'$ on account of their connectedness. Thus

$$H^*(G'/L') \otimes 1 \subset f_2^*(H^*(X_2)) = j_2'(H^*(G'/L_2')) \otimes j_2''(H^*(G''/L_2'')).$$

It follows then from (3.4) that $H^*(G'/L_1') \approx j_1'(H^*(G'/L_1'))$ can contain only elements of zero degree. But X_1 as well as G'/L_1' is orientable, and so $G' = L_1'$. From the equality $L_2' = L'$, we have $\dim G''/L_2'' \leqq \dim G''/L''$, and then we can use the same argument to show that $G'' = L_2''$, $L_1'' = L''$. It follows then

$$\begin{aligned} \dim X_1 + \dim X_2 &= \dim G''/L_1'' + \dim G'/L_2' \\ &= \dim G''/L'' + \dim G'/L' = \dim X. \end{aligned}$$

Proposition (4.8) then tells us that $X_1 = S^p$, $X_2 = S^p$ and $X = S^p \times S^q$.

(8.6) *Suppose that $n > 31$, and the Case B in* (8.3) *happens. If G is almost effective in S^n, then it is also almost effective on X_2.*

Proof. Let us write $G = G' \cdot G_0$ as local direct product of G' and G_0, where G_0 is the maximal connected normal subgroup of G acting trivially

on X_2 and G' a connected normal subgroup of G acting transitively and almost effectively on X_2. Evidently $G_0 \subset L_2$. Suppose that $Q, Q', N_2, M_s, M_s', \cdots$ have the same meaning as in (8.4). Since $L_2 = Q \cdot N_2$, the intersection $G_0 \cap N_2$ is normal in G_0, and therefore normal in G. By assumption, G acts almost effectively on S^n and so almost effectively on $X = G/L$. It follows that $G_0 \cap N_2$ must be finite, whence $G_0 \subset Q$. But Q is simple, and so either $G_0 = e$ or $G = Q$.

We shall show that $G_0 \neq Q$ by the method of contradiction. Suppose that $G_0 = Q$. Let us consider the local direct decompositions:

$$L_1 = M_s \cdot J \cdot N_1, \qquad L = M_s' \cdot J' \cdot N_1 = Q' \cdot N_2, \qquad L_2 = Q \cdot N_2.$$

Since $N_2 \cap G_0$ is finite and N_2 centralizes G_0, we can take our G' such that $N_2 \subset G'$. By assumption, $p + q = (n-1)/2 > 15$ and so we have, from (8.4), that $M_s' \subset N_2 \subset G'$. Here two possibilities arise. Let us discuss them separately.

(i) $M_s' \neq e$. The group M_s', being connected, must have positive dimension. Therefore the intersection $M_s \cap G'$ is a normal subgroup of M_s with positive dimension, and then from the simplicity of M_s, we have $M_s = M_s \cap G'$, i.e., $M_s \subset G'$. By our choice, M_s is transitive on L_1/L and therefore $L_1 = M_s L = (M_s N_2) \cdot Q$. Moreover, $L_2 = N_2 \cdot Q$, $L = N_2 \cdot Q'$ and $M_s N_2$, $N_2 \subset G'$ and $Q, Q' \subset G_0$. (8.5) then tells us that X_1 and X_2 are spheres and a contradiction is thus obtained.

(ii) $M_s' = e$. It follows that $p = 3$, $J = J' = e$ and

$$L_1 = M_s \cdot N_1, \quad L_2 = Q \cdot N_2, \quad L = N_1 = Q' \cdot N_2, \quad q > 12.$$

The subgroup M_s centralizes N_1 and hence centralizes Q'. But $Q/Q' = S^q$ and $q \neq 2$, and so the centralizer of Q' in Q is finite. It follows then from the connectedness of M_s and the local direct decomposition $G = G' \cdot G_0 = G' \cdot Q$ that $M_s \subset G'$. We have then

$$L_1 = (M_s N_2) \cdot Q', \quad L = N_2 \cdot Q', \quad L_2 = N_2 \cdot Q,$$

where $M_s N_2, N_2 \subset G'$ and $Q, Q' \subset G_0$. (8.5) then tells us that X_1, X_2 are spheres. This contradicts our assumption. Hence $G_0 = e$ and (8.6) is proved.

(8.7) *Suppose that $n > 31$, G acts almost effectively on S^n, and G' is a connected normal semi-simple subgroup of G transitive on X_2. If Case B* in (8.3) *happens, then G has a local direct decomposition $G = G' \cdot W$, where W is at most of rank 2 and is locally isomorphic with a subgroup of the centralizer of $L_2 \cap G'$ in G'. Moreover, $L_2 \cap W$ is finite.*

Proof. Since G is compact and connected, we can write $G = G' \cdot W$, where W is a connected normal subgroup of G. Up to a finite covering which does not affect any of our situation, we can assume, moreover, that $G' = G' \times W$. Let $\xi: G \to G'$, $\eta: G \to W$ be the projections, and $L_2' = L_2 \cap G'$. The connected group L_2 can be written in the form $L_2 = L_2' \cdot U$, where U is a certain connected and normal subgroup of L_2. Since G' is transitive on X_2 and $U \cap G'$ is finite, $W = \eta(U)$ and W is locally isomorphic with U.

The intersection $L_2 \cap W$ is a normal subgroup of L_2. Therefore, $L_2 \cap W$ is a normal subgroup of $\eta(L_2)$ which coincides with W. This tells us that $L_2 \cap W$ acts trivially on X_2. By (8.6), G is almost effective on X_2 and so $L_2 \cap W$ must be finite. This implies that $\xi(U)$ is locally isomorphic with U and hence with W. The subgroup U centralizes L_2' and therefore, $\xi(U)$ centralizes L_2'. From (8.3), we know that $L_2' \neq 0$ in G' and $r(L_2') = r(G') - 2$, where "r" denotes the rank. Moreover L_2' is semi-simple because G' is so. Hence the centralizer of L_2' is of rank 2 at most. Thus $r(W) = r(\xi(U)) \leqq 2$.

9. Some properties of G. In this section, we shall establish some properties of G which lead to the determination of the system $\{G, L, L_1, L_2\}$ when Case B in (8.3) happens. For this purpose, let us digress a little and prove two simple propositions.

(9.1) *Let W be a closed connected subgroup of a connected and compact Lie group V such that $r(W) = r(V) - s$, r denoting the rank. For any normal subgroup V' of V, $0 \leqq r(V') - r(V' \cap W) \leqq s$. Suppose W' to be a closed connected, normal non-abelian simple subgroup of W with $r(W') > s$. Then W' is contained in a certain simple normal subgroup of V.*

Proof. Consider the projection $\eta: V \to V/V'$. We have evidently

$$r(V/V') = r(V) - r(V'), r(\eta(W)) = r(W) - r(V' \cap W),$$
$$r(\eta(W)) \leqq r(V/V').$$

It follows that

$$\begin{aligned} r(V') - r(V' \cap W) &= r(V) - r(V/V') - r(W) + r(\eta(W)) \\ &\leqq r(V) - r(V/V') - r(W) + r(V/V') \\ &= r(V) - r(W) = s. \end{aligned}$$

To show the second part of (9.1), let us write $V = V_1 \cdot V_2 \cdot \cdots \cdot V_a$ as local direct product of simple normal subgroups, and

$$\xi_i : V \to V/V_1V_2\cdots V_{i-1}V_{i+1}\cdots V_a$$

be the projection $(i = 1, 2, \cdots, a)$. There exists at least one i, say 1, such that $\xi_i(W') \neq e$. Since W' is simple, $\xi_1(W')$ is locally isomorphic with W', and hence $r(\xi_1(W')) > s$. Let U be the connected component of the intersection $V_1 \cap W$. Since both U and W' are connected normal subgroups of W and W' is simple, either $W' \subset U$ or W' centralizes U. In the first alternative, $W' \subset V_1$ and our proposition is proved. Now let us show that the second alternative cannot happen. Suppose that W' centralizes U. Then $\xi_1(W')$ centralizes $\xi_1(U)$ in $\xi_1(V_1)$. But W' is simple and non-abelian and $r(\xi_1(W')) > s$, so the centralizer of $\xi_1(W')$ in $\xi_1(V_1)$ is of rank smaller than $r(\xi_1(V_1)) - s$. We have therefore $r(\xi_1(U)) < r(\xi(V_1)) - s$. Restricted to V_1, the homomorphism ξ_1 is a local isomorphism. It follows then $r(U) < r(V_1) - s$. On the other hand, we have from the first part of our Proposition (9.1) that $r(V_1) - r(U) = r(V_1) - r(V_1 \cap W) \leqq s$. A contradiction is thus obtained, and the proof is completed.

(9.2) *Let $V = V_1 \times V_2 \times \cdots \times V_a$ be a direct product of connected, compact simple Lie groups, and Y a connected normal subgroup of V with $r(V) = r(Y) + s$. If $r(V_i) > s$ for all i and Y is non-homologous to zero in V, then*

$$Y = (V_1 \cap Y) \times (V_2 \cap Y) \times \cdots \times (V_a \cap Y).$$

Proof. Let $Y_i = V_i \cap Y$. From the first part of (9.1), it follows that $r(Y_i) \geqq r(V_i) - s \geqq 1$. Since $Y \neq 0$ in V, the number of simple local direct factors is less than a, and therefore the normal subgroups $Y_1, Y_2, \cdots, Y_a$ of Y must exhaust all its direct factors. Hence $Y = Y_1 \times Y_2 \times \cdots \times Y_a$.

(9.3) *Suppose that $n > 31$ and Case B in (8.3) happens. Then G has a semi-simple connected normal subgroup G' such that G' is transitive on X_2 and that G' has at most two simple director factors.*

Proof. From (8.3), we know that $h_2^* : H^*(G/L_2) \to H^*(G)$ is injective and its image is generated by two primitive elements x, y of degrees p and $p+q$ respectively. Let us write G as the local direct product $G_1 \cdot G_2 \cdots G_a$ of connected simple normal subgroups. There exists at least a simple factor, say G_1, such that, under the homomorphism $j_1^* : H^*(G) \to H^*(G_1)$ induced by the injection $j_1 : G_1 \to G$, the image $x_1 = j_1^*(x)$ of x is different from zero. Here two alternatives arise. Let us discuss them separately.

(i) Suppose that $y_1 = j_1^*(y) \neq 0$. Consider the commutative diagram

$$\begin{array}{ccc} & \overset{h_2}{\longleftarrow} & \\ G/L_2 & & G \\ \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle j_1} \\ & \overset{k}{\longleftarrow} & \\ G_1/(G_1 \cap L_2) & & G_1 \end{array}$$

and its induced diagram on the real cohomology algebras

$$\begin{array}{ccc} & \overset{h_2^*}{\longrightarrow} & \\ H^*(G/L_2) & & H^*(G) \\ \downarrow{\scriptstyle l^*} & & \downarrow{\scriptstyle j_1^*} \\ & \overset{k^*}{\longrightarrow} & \\ H^*(G_1/(G_1 \cap L_2)) & & H^*(G_1), \end{array}$$

where k is the projection and l is the map induced by the inclusion $G_1 \cap L_2 \subset L_2$. Thus $k^*(H^*(G_1/(G_1 \cap L_2))$ contains two non-zero primitive elements x_1, y_1 of degrees p, $p+q$ respectively. It follows that

$$\dim(G_1/(G_1 \cap L_2)) \geqq 2p+q = \dim G/L_2.$$

On the other hand, $G_1/(G_1 \cap L_2)$ can be regarded as the orbit of G_1 when it acts on G/L_2. Therefore G_1 is transitive on X_2. (9.3) is proved in this case.

(ii) Suppose that $j_1^*(y) = 0$. There exists another simple factor, say G_2, such that $j_2^*(y) \neq 0$, where j_2 is the injection: $G_2 \to G$. Just as before, but replacing G_1 by $G_1 \cdot G_2$, we can show that $G_1 \cdot G_2$ is transitive on X_2. This completes the proof.

(9.4) *Suppose that $n > 31$ and Case B in* (8.3) *happens. Then G cannot be simple.*

Proof. Suppose G to be simple. Let $Q, Q', N_2, M, M_s', \cdots$ have the same meaning as in (8.4). Then $L_2 = Q \cdot N_2$, where Q acts transitively and almost effectively on the q-sphere L_2/L. Since $L_2 \not\sim 0$ in G and G is simple, L_2 must be simple and hence $N_2 = e$. By (8.4), $M_s' \subset N_2 = e$, and then $p = 3$. But we know that $p+q = (n-1)/2 > 15$. Therefore $q > 12$ and Q must be a B_m with $m = q/2 > 6$. Thus the simple group G has the properties: (a) $B_m \not\sim 0$ in G, (b) $r(G) = m+2$, and (c)

$$\mathfrak{P}(G) = \mathfrak{P}(B_m)(1+t^3)(1+t^{2m+3}), \quad m > 6,$$

where $\mathfrak{P}$ denotes the Poincaré polynomial with t as the indeterminate. By a direct enumeration of simple groups, we find that no such G exists. Our proposition is thus proved.

(9.5) *Suppose that $n > 31$ and Case B in (8.3) happens. If G has no proper normal subgroup transitive on X, then G is the local direct product of two simple Lie groups.*

Proof. Without loss of generality, we can assume that G is the direct product of simple Lie groups. On account of (9.4), it suffices to show that G has at most two simple factors. Let G' be the normal subgroup of G as contructed in (9.3), and let us write $G = G' \times W$. Since Q is transitive on L_2/L and G' transitive on X_2, we have $QL = L_2$, $G'QL = G'L_2 = G$. This tells us that $G'Q$ is transitive on X. Let $\eta: G \to W$ be the projection. Then $\eta(L_2) = W$ and $G'Q = \eta^{-1}(\eta(Q))$. Since Q is normal in L_2, $\eta(Q)$ is normal in W, whence $G'Q$ is normal in G. Therefore $G = G'Q$ on account of our assumption. If $Q \subset G'$, then $G = G'$ and by (9.3), G has at most two simple factors. If G' is simple, then G is locally isomorphic with the direct product $G' \times Q$ which has two simple factors. Thus we have proved our proposition when either $Q \subset G'$ or G' is simple.

In fact, the above two cases are the only alternatives. Let us show this by the method of contradiction. Suppose that $Q \not\subset G'$ and G' is not simple. By (9.3), we can write $G' = V_1 \times V_2$, where V_1 and V_2 are simple and none of them is transitive on X_2. Let $L_2' = L_2 \cap G'$. Then $X_2 = G'/L_2'$. Since X_2 is simply-connected and G' connected, the subgroup L_2' must be connected. The group Q being simple and $Q \not\subset G'$, it follows that $\eta(Q)$ is locally isomorphic with Q. From (8.7), we know that $r(W) \leqq 2$ and then

$$r(Q) = r(\eta(Q)) \leqq r(W) \leqq 2.$$

Let $\xi: G \to G'$, $\xi_1: G \to V_1$, $\xi_2: G \to V_2$ be the projections. From (8.7), $\xi(L_2)$ is locally isomorphic with L_2. Now we find it convenient to divide our further discussions into two caeses.

Case I. $r(Q) = 2$. Then $r(W) = 2$ and

$$r(G') = r(G) - r(W) = r(G) - 2 = r(L_2) = r(\xi(L_2)).$$

The subgroup $\xi(L_2)$, being of the same rank as G', can be written as the direct product of a subgroup in V_1 and a subgroup in V_2 [14, p. 927]. Since $L_2 \neq 0$ in G, $L_2' \neq 0$ in G'. Hence L_2' is a connected, normal semi-simple subgroup of L_2 and hence we can write $L_2' = Y_1 \times Y_2$, where $Y_1 \subset V_1$, $Y_2 \subset V_2$. Evidently, Y_1, Y_2 are connected. Moreover, from the facts $r(L_2') = r(G') - 2$ and $L_2' \neq 0$ in G', it follows that

$$Y_1 \neq 0 \text{ in } V_1, \quad Y_2 \neq 0 \text{ in } V_2, \; r(Y_1) + r(Y_2) = r(V_1) + r(V_2) - 2.$$

By (8.7), $\xi(Q) \neq e$, and therefore either $\xi_1(Q)$ or $\xi_2(Q)$ is non-trivial, say $\xi_1(Q) \neq e$. Q and Y_1 are two connected normal subgroups of L_2 with $Q \cap Y_1$ finite. It follows that Q and Y_1, and hence $\xi_1(Q)$ and Y_1 centralize each other. But $\xi_1(Q)$ is simple and of rank 2, so

$$r(V_1) \geqq r(Y_1 \cdot \xi_1(Q)) = r(Y_1) + r(\xi_1(Q)) = r(Y_1) + 2.$$

We have therefore $r(Y_2) = V_2$ and then $Y_2 = V_2$. This contradicts the fact that V_1 is not transitive on X_2. In other words, this case cannot happen.

Case II. $r(Q) = 1$. From the construction of V_1, V_2 in (9.3), we know that one of V_1, V_2 has a primitive exponent[7] p while the other a primitive exponent $p+q$. Since $r(Q) = 1$ and $2(p+q) = n > 31$, we have $q = 2$, $p > 13$, whence

$$r(V_1) \geqq 4, \qquad r(V_2) \geqq 4.$$

Let $Y_1 = L_2' \cap V_1 = L_2 \cap V_1$, $Y_2 = L_2' \cap V_2 = L_2 \cap V_2$. Since $L_2' \neq 0$ in G' and $r(L_2') = r(G') - 2$, it follows from (9.2) that $L_2' = Y_1 \times Y_2$, whence $X_2 = G'/L_2' = (V_1/Y_1) \times (V_2/Y_2)$. We know that X_2 is simply connected, neither V_1 nor V_2 is transitive on X_2 and $H^*(X_2) \approx H^*(S^p \times S^{p+q})$. It follows that V_1/Y_1 and V_2/Y_2 are simply connected homology spheres with real coefficients. One of them has dimension p while the other dimension $p+2$. The integers p, $p+2$ cannot be both congruent to 3 mod 4. To be precise, let us take $\dim V_1/Y_1 \not\equiv 3 \pmod 4$. Y_1 and Q are two connected normal subgroups of L_2 with $Q \cap Y_1$ finite. It follows that Y_1 and Q, and hence Y_1 and $\xi_1(Q)$, centralize each other. But $\xi_1(Q)$ is simple and so $Y_1 \cap \xi_1(Q)$ is a finite central subgroup of $\xi_1(Q)$. Now let use consider the fibring $V_1/Y_1 \to V_1/(Y_1 \cdot \xi_1(Q))$ whose fibres are homeomorphic with $F = \xi_1(Q)/Y_1 \cap \xi_1(Q)$. Since $r(Q) = 1$, F is either a point, a 3-sphere or a 3-dimensional real projective space. We know that V_1/Y_1 is a simply-connected real homology sphere of dimension $\not\equiv 3 \pmod 4$. It cannot have any orientable fibre decomposition with the 3-dimensional real homology sphere as the fibre. Hence F must be a point and $\xi_1(Q) = e$, or in other words, $Q \subset V_2 \times W$. Let us denote $V_2 \times W$ by W', and let $Q', N_2, M_8, \cdots$ have the same meaning as in (8.4). Since $G = G'Q$, $L_2 = L_2' \cdot Q$ and $L = L_2' \cdot Q'$, we have

$$\begin{gathered} G = V_1 \times W',\ L_2 = Y_1 \times Y_2 Q,\ L = Y_1 \times Y_2 Q',\ Y_1 \subset V_1 \\ Y_2 Q' \subset Y_2 Q \subset W'. \end{gathered} \tag{9.6}$$

[7] The degree of a primitive element of the real cohomology of a compact Lie group is called a primitive exponent of that group.

Since $p > 13$, $r(M_s) \geqq 4$, $r(L_1) = r(G) - 1$, it follows from (9.1) that M_s belongs to a simple factor of G. Therefore, either $M_s \subset V_1$ or $M_s \subset W'$, and then

$$L_1 = M_s L \subset (L_1 \cap V_1) \times (L_1 \cap W') \subset L_1. \tag{9.7}$$

Taking account of (9.6) and (9.7), we know from (8.5) that X_1, X_2 are spheres. A contradiction is thus obtained and (9.5) is proved.

(9.8) *Suppose that $n > 31$ and Case B in (8.3) happens. Then $q \leqq 4$ and the group G has a simple normal subgroup G' transition on X_2.*

Proof. Let G^* be a minimum normal subgroup of G transitive on X. Then G and G^* have the same orbits on S^n, and every connected normal subgroup of G^* is a normal subgroup of G. Therefore, without affecting any of our situation, we can simply take G to be G^*. Furthermore, without loss of generality, we can assume that G is a direct product of simple Lie groups. By (9.5), $G = V_1 \times V_2$, where V_1, V_2 are simple. Since X is simply-connected, the semi-simple part of G is transitive on X [15, p. 4], and so G must be semi-simple and V_1, V_2 non-abelian. Now let us divide our discussions into two cases.

Case I. $p = 3$. Then $q > 12$, $Q = B_m$, where $m = q/2 > 6$. The subgroup $L_2 = N_2 \cdot Q$ is non-homologous to zero in $V_1 \times V_2$, and

$$\mathfrak{P}(V_1)\mathfrak{P}(V_2) = \mathfrak{P}(Q)\mathfrak{P}(N_2)(1 + t^3)(1 + t^{3+q}),$$

where $\mathfrak{P}$ denotes the Poincaré polynomial. It follows that N_2 has vanishing first and third Betti number, and hence $N_2 = e$. (8.4) then tells us that $M_s' = e$, and then

$$J \approx J' = e,\ M_s = B_1,\ L_2 = Q,\ L_1 = M_s \cdot Q',\ L = Q'.$$

Taking account of the facts $r(Q) = r(L_2) = r(G) - 2$ and $r(Q) > 2$, we have from (9.1) that Q belongs to a simple normal subgroup of G, say $Q \subset V_2$. Therefore, $X_2 = V_1 \times (V_2/Q)$. By (8.6), G is almost effective on X_2 whence $V_2 \neq Q$. Since V_1 is simple and non-trivial, it follows from the equality $\mathfrak{P}(X_2) = (1 + t^3)(1 + t^{3+q})$ that $\mathfrak{P}(V_2/Q) = 1 + t^{3+q}$. Hence

$$\mathfrak{P}(V_2) = \mathfrak{P}(Q)(1 + t^{3+q}) = \mathfrak{P}(B_m)(1 + t^{3+q}), \quad m = q/2 > 6.$$

An enumeration of simple groups tells us that no such simple V_1 exists. In other words, the case by which $p = 3$ cannot happen.

Case II. $p > 3$. Since $L_2 = Q \cdot N_2$ is non-homologous to zero in $V_1 \times V_2$,

N_2 must be simple. By (8.4), M_s' is a normal subgroup of N_2. Here $p > 3$, $M_s' \neq e$ and so $M_s' = N_2$, whence

$$(9.9) \qquad L_2 = Q \cdot M_s', \; L_1 = M_s \cdot J \cdot N_1, \; L = Q' \cdot M_s', \; Q' = J' \cdot N_1.$$

Case IIa. Suppose that $q > 4$. Then Q' is a simple group of rank greater than or equal to two. Since $r(J') \leqq 1$, the last equality in (9.9) tells us that $J' = e$, and then $J = e$, $Q' = N_1$ and

$$L_2 = Q \cdot M_s', \; L_1 = Q' \cdot M_s, \; L = Q' \cdot M_s'.$$

From the facts $p > 3$, $q > 4$, we have $r(M_s) > 1$, $r(Q') > 1$. Moreover, $r(L_1) = r(G) - 1$ by (8.3). It follows then from (9.1) that both M_s and Q' belong to certain simple normal subgroups of G. Suppose that M_s, Q' belong to a same simple normal subgroup, say V_1. Since Q is simple and $Q' \subset Q$, Q also belongs to V_1, and then $L_2 = Q \cdot M_s' \subset V_1$. The last inclusion would imply that $L_2 \frown 0$ in V_1 which is impossible as L_2 has two simple normal factors. Therefore, M_s and Q' belong to different simple normal subgroups of G, say $M_s \subset V_1$, $Q' \subset V_2$. It follows that $Q \subset V_2$, and then (8.5) tells us that X_1 and X_2 are spheres. This contradicts our hypothesis. In other words, q cannot be greater than 4.

Case IIb. Suppose that $q \leqq 4$. We shall prove the second part of our proposition by method of contradiction. Suppose that G has no simple normal subgroup transitive on X_2. From the proof of (9.3), we know that one of V_1, V_2 has a primitive exponent p while the other has a primitive exponent $p + q$. Since $q \leqq 4$, $n > 31$, we have $p > 11$ and

$$r(V_1) \geqq 3, \qquad r(V_2) \geqq 3.$$

Taking account of the fact $r(L_2) = r(G) - 2$, we know from (9.1) that

$$r(V_1 \cap L_2) \geqq 1, \qquad r(V_2 \cap L_2) \geqq 1.$$

But $V_1 \cap L_2$, $V_2 \cap L_2$ are normal subgroups of L_2, and L_2 has only two proper normal subgroups. Therefore, up to a change of notation, we can assume that $M_s' \subset V_1$, $Q \subset V_2$. Since $M_s' \neq e$ and M_s is simple, it follows that $M_s \subset V_1$ and then

$$L_1 = M_s L = M_s \cdot Q', \; L_2 = M_s' \cdot Q, \; L = M_s' \cdot Q',$$

where $Q' \subset Q \subset V_2$, $M_s' \subset M_s \subset V_1$. (8.5) then tells us that X_1, X_2 are spheres. A contradiction is thus obtained. In other words, G has a simple normal subgroup transitive on X_2.

10. Determination of G when p is odd and q is even. Now we are in a position to determine the group G and the orbits X, X_1, X_2 for even q and odd $p>1$. Let $Sp(r)$ denote the group of all $(r\times r)$ quaternion symplectic matrices. If a subgroup V of $Sp(r)$ is conjugate to the subgroup of all matrices $q=(q_{ij})\in Sp(r)$ with $q_{11}=1$, then we shall call V an $Sp(r-1)$ imbedded in $Sp(r)$ in the canonical position. Canonical imbedding for unitary groups, special unitary groups and the orthogonal groups is defined similarly.

(10.1) *Suppose that $n>31$ and Case B in (8.3) happens. If $q=4$, then X_2 is the quaternion Stiefel manifold $Sp(r)/Sp(r-2)$, $X=X_2\times S^4$, and X_1 is an S^{p+4}-bundle over S^4, where $p=4r-5$. If the group G is almost effective on S^n, then up to a finite covering of G*

$$G=Sp(r)\times Sp(2),\ Q\approx Sp(2),\ Q\cap Sp(r)=Q\cap Sp(2)=e,$$
$$M_s\approx Sp(r-1),\ M_s'\approx Sp(r-2),$$

where the imbeddings $M_s'\subset M_s$, and $M_s\subset Sp(r)$ are in the canonical manner.

Proof. Up to a finite covering, we can assume that G is a direct product of simply-connected compact simple Lie groups and circle groups. Now let $L, L_1, L_2, Q, Q', M_s, M_s', \cdots$ have the same meaning as before. Since $n>31$, $q=4$ and Case B in (8.3) happens, we know that $p>11$ and $Q=C_2$. From (9.8), G has a simple normal subgroup G' transitive on X_2, and from (8.7), we can write $G=G'\times W$. The subgroup Q cannot be contained in G' for, if so, G' would be transitive on X and would have the same orbits as G which contradicts (9.4). Therefore, $Q\cap G'$ is finite. Let $\eta: G\to W$ be the projection. Then $\eta(Q)$ is locally isomorphic with Q and hence a C_2. By (8.7), W is a compact Lie group of rank not greater than 2. We know that there is no connected group of rank 2 which can contain C_2 as a proper subgroup, and so $\eta(Q)=W$. But the fundamental group of G has no torsion. It follows then W is simply-connected, and then $W=Sp(2)$, $W\approx Q$, $Q\cap G'=e$. Since G is the direct product of two simple non-abelian groups, all the arguments in Case II of (9.8) hold, and then

$$L_2=M_s'\cdot Q,\ L_1=M_s\cdot Q',\ L=M_s'\cdot Q',\ Q'=J'\cdot N_1. \tag{10.2}$$

The group $\eta(M_s')$ centralizes $\eta(Q)$ and $\eta(Q)=W=Sp(2)$. It follows that $\eta(M_s')=e$, or what is the same, $M_s'\subset G'$, whence $M_s\subset G'$. Therefore M_s' is the identity component of $L_2\cap G'$. But $X_2=G/L_2=G'/(L_2\cap G')$ is simply connected, and so $L_2\cap G'$ is connected. We have then $M_s'=L_2\cap G'$ and $X_2=G'/M_s'$. Let $\xi: G\to G'$ be the projection. By (8.7), L_2 is locally isomorphic with $\xi(L_2)=M_s'\cdot\xi(Q)$, and, from (8.3),

$$r(G') = r(G) - r(W) = r(G) - 2 = r(L_2) = r(\xi(L_2)).$$

We know thus that the pair $\{G', M_s'\}$ has the following five properties:

(i) G', M_s' are simple groups,

(ii) M_s' is non-homologous to zero in G',

(iii) $M_s/M_s' = S^p$, $p = \text{odd} > 11$,

(iv) $\mathfrak{P}(G'/M_s') = \mathfrak{P}(X_2) = (1+t^p)(1+t^{p+4})$, and

(v) G' has a subgroup $M_s' \cdot \xi(Q)$ having the same rank as G', where $\xi(Q)$ is a C_2.

By a direct enumeration of simple groups, we find that

$$G' = C_r,\ M_s' = C_{r-2},\ M_s = C_{r-1},\ p = 4r-5.$$

Since $\pi_1(G)$ has no torsion, $G' = Sp(r)$, $G = Sp(r) \times Sp(2)$.

Now let us discuss the position of M_s in G', and the position of M_s' in M_s. We recall that

$$L_1 = M_s \cdot J \cdot N_1,\ L = M_s' \cdot J \cdot N_1,\ Q' = J' \cdot N_1 \approx C_1 \times C_1,$$
$$J' = M_s \cdot J,\ r(J) = r(J') \leqq 1.$$

Since $r(G) = r(L_1) + 1$,

$$r(L_1 \cap W) = r(L_1) - r(\xi(L_1)) \geqq r(L_1) - r(G') = r(L_1) - r(G) + 2 = 1.$$

On the other hand, $M_s \subset G'$, $N_1 \cap W \subset Q \cap W = e$. It follows that J is the identity component of $L_1 \cap W$ and $r(J) = 1$, $N_1 \approx C_1$. Therefore, $r(L_1 \cap W) = 1$, and the subgroup $\xi(L_1) = M_s \cdot \xi(N_1)$ has the same rank as G'. Up to an automorphism of $Sp(r)$, there is only one connected subgroup of the class C_{r-1} which is a normal subgroup of a subgroup of maximum rank in $Sp(r)$ [14]. Hence M_s is an $Sp(r-1)$ imbedded in $G' = Sp(r)$ in the canonical position. We know that M_s/M_s' is a p-sphere. Up to an automorphism of M_s, there is only one such subgroup M_s' [17].[8] Therefore, M_s' is a $Sp(r-2)$ imbedded in $M_s = Sp(r-1)$ in the canonical position.

We are now in a position to determine the orbits X, X_1, X_2. Obviously, $X_2 = G'/M_s' = Sp(r)/Sp(r-2)$ is the quaternion Stiefel manifold. To see X, let us consider the map

$$\theta\colon (G'/M_s') \times (Q/Q') \to G/L = X$$

given by $\theta(gM_s', qQ') = gqL$, where $g \in G'$, $q \in Q$. We can easily verify that

[8] This is equivalent to saying that M_s acts on the p-sphere M_s/M_s' orthogonally.

this mapping θ is well defined. From the fact that $G' \cap Q = e$, θ must be injective. Moreover, $(G'/M_s') \times (Q/Q')$ and X have the same dimension, and so θ is bijective. This tells us that X is homeomorphic with $X_2 \times S^4$. To see X_1, let us consider G' to be a transformation group of $X_1 = G/L_1$. Since G' is normal in G, the G'-orbit decomposition of X_1 is a fibre decomposition. The fibres (i.e., orbits under G') are all homeomorphic with

$$G'/(G' \cap L_1) = G'/M_s = Sp(r)/Sp(r-1) = S^{4r-1} = S^{p+4}$$

whereas the base space (i.e., the space of orbits) is

$$G/(G'L_1) = W(W \cap L_1) = W/\eta(Q') = Q/Q' = S^4.$$

Therefore X_1 is an S^{p+4}-bundle over S^4. This completes the proof of (10.1).

Remark. G does not act effectively on S^n. There is a central subgroup I_2 of order 2 which is contained in neither direct factor of G. The factor group G/I_2 acts effectively on S^n.

(10.3) *Let G, $\bar{G}$ be two compact connected effective differentiable transformation groups of S^n with $(n-1)$-dimensional orbit. Suppose that $n > 31$, p is odd, $q = 4$, and their special orbits are not spheres. Let L, L_1, L_2 and $\bar{L}$, $\bar{L}_1$, $\bar{L}_2$ be the isotropic subgroups of G and $\bar{G}$ respectively defined as above. Then there is an isomorphism of G onto $\bar{G}$ carrying L, L_1, L_2 into $\bar{L}$, $\bar{L}_1$, $\bar{L}_2$ respectively.*

Proof. The universal covering groups of G, $\bar{G}$ act on S^n in the natural manner, and have the same orbits as G, $\bar{G}$ respectively. We can readily see that if these two covering groups satisfy the conclusion of (10.3), so do G and $\bar{G}$. Hence it is sufficient for us to prove our proposition for the case that G and $\bar{G}$ are simply-connected. Of course, G and $\bar{G}$ are now no longer effective. From (10.1),

$$G = \bar{G} = Sp(r) \times Sp(2).$$

Suppose that M_s, M_s', Q, Q' are subgroups of G defined by (8.4), and $\bar{M}_s$, $\bar{M}_s'$, $\bar{Q}$, $\bar{Q}'$ are subgroups of $\bar{G}$ similarly defined. From (10.1), the pair (M_s, M_s') and $(\bar{M}_s, \bar{M}_s')$ are conjugate in $G' = Sp(r)$, and thus we can assume

$$M_s = \bar{M}_s = Sp(r-1), \qquad M_s = \bar{M}_s' = Sp(r-2),$$

where the imbeddings $Sp(r-2) \subset Sp(r-1)$, $Sp(r-1) \subset Sp(r)$ are in the canonical manner. Let C be the identity component of the centralizer of M_s' in $Sp(r)$. Then $C \approx Sp(2)$. Since $C \times Sp(2)$ is the identity component of the centralizer of M_s' in G, $Q \subset C \times Sp(2)$. From (10.1), $Q \cap C = Q \cap Sp(2)$

$= e$. Therefore, Q is a diagonal of $C \times Sp(2)$. Up to an automorphism of $Sp(2)$, there is only one diagonal of $C \times Sp(2)$. Since $\bar{Q}$ is also a diagonal of $C \times Sp(2)$ and this automorphism of $Sp(2)$ does not affect $Sp(r)$, we can further assume that $Q = \bar{Q}$.

We know from (10.2) that Q' and M_s centralize each other. Since M_s is an $Sp(r-1)$ imbedded in $Sp(r)$ in the canonical position and Q is a diagonal of $C \times Sp(2)$, we can verify readily that Q' is the maximum connected subgroup of Q which centralizes M_s. Therefore, Q' is completely determined by M_s and Q. From the equalities $M_s = \bar{M}_s$, $Q = \bar{Q}$, it follows that $Q' = \bar{Q}'$. (10.2) then tells us that $L = \bar{L}$, $L_1 = \bar{L}_1$, $L_2 = \bar{L}_2$. Proposition (10.3) is thus proved.

(10.4) *Suppose that $n > 31$, $q = 2$, p is odd, and Case B in* (8.3) *happens. Then X_2 is the complex Stiefel manifold $SU(r)/SU(r-2)$, $X = X_2 \times S^2$, and X_1 is an S^{p+2}-bundle over S^2, where $p = 2r-3$. If, moreover, G is semi-simple, simply-connected and almost effective on S^n, then*

$$G = SU(r) \times SU(2),\ Q \approx SU(2),\ Q \cap SU(r) = Q \cap SU(2) = 2,$$

$$M_s = SU(r-1),\ M_s' = SU(r-2),$$

$$L_2 = M_s' \cdot Q,\ L_1 = M_s \cdot Q',\ L = M_s' \cdot Q',$$

and the imbeddings $M_s \subset SU(r)$, $M_s' \subset M_s$ are canonical.

Proof. The proof is similar to that of (10.1).

(10.5) *Suppose that $n > 31$, $q = 2$, p is odd, and Case B in* (8.3) *happens. If G acts almost effectively on S^n, then G is either semi-simple or a local direct product of a circle group and a semi-simple group.*

Proof. Let us write G in the form of local direct product $G^* \cdot W$, where G^* denotes the semi-simple part and W the identity component of the center of G. Since X, X_1, X_2 are all simply-connected, G and G^* have the same orbits on S^n [15, p. 4]. By (8.7), W is locally isomorphic with a subgroup of the centralizer of $L_2 \cap G^*$ in G^*. On the other hand, by applying (10.4) to G^*, we know that the centralizer of $L_2 \cap G^*$ in G^* is one-dimensional. Therefore W is either trivial or the circle group, and (10.5) is thus proved.

(10.6) *Let G, $\bar{G}$ be two compact connected effective differentiable transformation groups of S^n with an $(n-1)$-dimensional orbit. Suppose that $n > 31$, $q = 2$, p is odd, and their special orbits are not spheres. Let L, L_1, L_2 and $\bar{L}$, $\bar{L}_1$, $\bar{L}_2$ be the isotropic subgroups of G and $\bar{G}$ respectively defined as*

above. If G and $\bar{G}$ are locally isomorphic, then there is an isomorphism of G onto $\bar{G}$ carrying L, L_1, L_2 into $\bar{L}$, $\bar{L}_1$, $\bar{L}_2$ respectively.

Proof. The groups G, $\bar{G}$ can be semi-simple or a local direct product of a circle group and a semi-simple group. We first prove the semi-simple case and then use the result to prove the other case. The argument is similar to the argument used in the proof of (10.3), and we omit the details.

Remark. The assumptions are the same as in (10.6). If G is semi-simple, then G is isomorphic with $SU(r)\times SU(2)$ or $(SU(r)\times SU(2))/I_2$ according as $p\equiv 3$ or $1 \pmod 4$, where I_2 denotes a central subgroup of $SU(r)\times SU(2)$ which belongs to neither factor. If G is not semi-simple, then G is isomorphic with the factor group $(U(r)\times U(2))/T$, where $U(k)$ denotes the unitary group in k variables and T denotes a closed one-dimensional central subgroup of $U(r)\times U(2)$ which intersects neither of the factors.

11. The case by which p or q equals 1. When p or q is equal to 1, the isotropic subgroups L, L_1, L_2 may not be connected. Our previous method to study the orbits of G has to be modified. In this section, we shall first compare the cohomology of X, X_1, X_2 with their covering spaces and then determine them.

(11.1) *Let L^0 denote the identity component of L. Then* $\mathrm{Ad}\, L = \mathrm{Ad}\, L^0$, *where "*Ad*" denotes the linear adjoint representation on its Lie algebra.*

Proof. If $p>1$ and $q>1$, then $L^0 = L$, and then this becomes trivial. Now let us assume $p=1$. Suppose $q>1$. Since X is an S^q-bundle over X_2, the homomorphism $f_2^{\#}\colon \pi_1(X)\to\pi_1(X_2)$ is bijective. By (4.3), X_1 must be simply-connected, and hence L_1 is connected. But L_1/L is a circle, and so $L_1 = T\cdot L^0$, where T is a one-dimensional central subgroup of L_1. From the fact that $L\subset L_1 = T\cdot L^0$, it follows immediately that $\mathrm{Ad}\, L = \mathrm{Ad}\, L^0$. Thus (11.1) is proved when $q>1$.

Now suppose that $p=q=1$. Then L_1/L, L_2/L are circle groups, and therefore $L_i = T_i\cdot L$, where T_i is a circle group belonging to the center of L_i $(i=1,2)$. Let $L_i{}^0$ denote the identity component of L_i. Since, for $i=1,2$, $L_i{}^0\cap L\subset L_i{}^0 = T_i\cdot L^0$, we have

$$\mathrm{Ad}(L_1{}^0\cap L) = \mathrm{Ad}\, L^0, \qquad \mathrm{Ad}(L_2{}^0\cap L) = \mathrm{ad}\, L^0.$$

Proposition (4.9) then implies that $\mathrm{Ad}\, L = \mathrm{Ad}\, L^0$. The proof is thus completed.

(11.2) *The homomorphism* $\theta^*: H^*(G/L) \to H^*(G/L^0)$ *induced by the covering map* $\theta: G/L^0 \to G/L$ *is bijective.*

Proof. Since the coefficient field is the field of real numbers, θ^* is injective and the image of θ^* consists of the elements of $H^*(G/L^0)$ invariant under the group Σ of covering transformations. Let us recall that, for any Lie group Q (not necessarily connected), $I(Q)$ denotes the algebra of symmetric multilinear forms over the Lie algebra of Q invariant under $\mathrm{Ad}\, Q$. From Cartan's results [4], we know that there is a canonical isomorphism between $H^*(G/L^0)$ and $H^*(I(L^0) \otimes H^*(G))$, where the differential operator over $I(L^0) \otimes H^*(G)$ depends only on L^0, G and the position of L^0 in G. Let us identify $H^*(G/L^0)$ with $H^*(I(L^0) \otimes H^*(G))$. Since the group Σ of covering transformations is finite, we see immediately that $H^*(I(L) \otimes H^*(G))$ consists of all the Σ-invariant elements in $H^*(I(L^0) \otimes H^*(G))$. Therefore, $\theta^*(H^*(G/L)) = H^*(I(L) \otimes H^*(G))$. But from (11.1), it follows $I(L) = I(L^0)$, and hence θ^* is surjective. (11.2) is thus proved.

(11.3) *Let* $e_i^*: H^*(G/L_i^0) \to H^*(G/L^0)$, $\theta_i^*: H^*(G/L_i) \to H^*(G/L_i^0)$ *be the homomorphisms induced by the natural maps:*

$$G/L^0 \to G/L_i^0, \quad G/L_i^0 \to G/L_i, \quad (i = 1, 2).$$

Then

$$H^*(G/L_i^0) = \theta_i^*(H^*(G/L_i)) \oplus \ker e_i^*.$$

Proof. Let l be an element of G which normalizes L^0 and L_i^0. We define $\phi(l): G/L^0 \to G/L^0$ to be the homeomorphism $gL \to glL$, and $\phi_i(l): G/L_i^0 \to G/L_i^0$ the homeomorphism $gL_i^0 \to glL_i^0$, $(g \in G)$. Then the diagram

$$\begin{array}{ccc} H^*(G/L^0) & \xrightarrow{\phi(l)^*} & H^*(G/L^0) \\ \uparrow{\scriptstyle e_i^*} & & \uparrow{\scriptstyle e_i^*} \\ H^*(G/L_i^0) & \xrightarrow{\phi_i(l)^*} & H^*(G/L_i^0) \end{array}$$

is commutative. From (11.2), $\phi(l)^*$ is the identity for all $l \in L$. Since L is compact, we see readily from the diagram that $e_i^*(J) = e_i^*(H^*(G/L_i^0))$, where

$$J = \{u: \phi_i(l)^*(u) = u, l \in L\}$$

is the set of all $\phi_i(L)^*$-invariant elements in $H^*(G/L_i^0)$. But $L_i = LL_i^0$, and so J is also the set of all $\phi_i(L_i)$-invariant elements in $H^*(G/L_i^0)$. On the other hand, G/L_i^0 is a covering space of G/L_i with $\phi_i(L_i)$ as its group of covering transformations. Therefore

(11.4) $$J = \theta_i^*(H^*(G/L_i)).$$

From the commutative diagram

$$\begin{array}{ccc} H^*(G/L) & \xrightarrow{\theta^*} & H^*(G/L^0) \\ {\scriptstyle f_i^*}\uparrow & & \uparrow{\scriptstyle e_i^*} \\ H^*(G/L_i) & \xrightarrow{\theta_i^*} & H^*(G/L_i^0) \end{array}$$

we know that, when restricted to J, the homomorphism e_i^* is injective. In other words, $J \cap \ker e_i^* = 0$. Our proposition then follows from (11.4) and the equality $e_i^*(J) = e_i^*(H^*(G/L_i^0))$ which we have established.

(11.5) *If X_2 is orientable, then the homomorphism θ_2^*: $H^*(G/L_2) \to H^*(G/L_2^0)$ induced by the covering map: $G/L_2^0 \to G/L_2$ is bijective. If X_2 is not orientable, then*

$$\mathfrak{P}(X) = (1 + t^{2q+1})\mathfrak{P}(X_2), \qquad \mathfrak{P}(G/L_2^0) = (1 + t^{q+1})\mathfrak{P}(X_2),$$

where "$\mathfrak{P}$" denotes the Poincaré polynomial with t as the indeterminate.

Proof. G/L^0 is an orientable S^q-bundle over G/L_2^0. Therefore we have the Gysin-Chern-Spanier sequence with real coefficients:

$$H^*(G/L_2^0) \xrightarrow{e_2^*} H^*(G/L^0) \xrightarrow{\partial} H^*(G/L_2^0) \xrightarrow{\omega} H^*(G/L_2^0) \xrightarrow{e_2^*} H^*(G/L^0),$$

where e_2^* is degree preserving, while ∂ lowers the degree by q and ω raises the degree by $q+1$. For any graded linear space E, let us denote by $\mathfrak{P}(E)$ the Poincaré polynomial of E with t as the indeterminate. Then from the above exact sequence, we have

$$\mathfrak{P}(G/L^0) = \mathfrak{P}(\operatorname{im} e_2^*) + t^q\mathfrak{P}(\operatorname{im}\partial), \qquad \mathfrak{P}(G/L_2^0) = \mathfrak{P}(\operatorname{im}\partial) + \mathfrak{P}(\operatorname{im}\omega)/t^{q+1},$$
$$\mathfrak{P}(G/L_2^0) = \mathfrak{P}(\operatorname{im}\omega) + \mathfrak{P}(\operatorname{im} e_2^*).$$

Let $J = \theta_2^*(H^*(G/L_2))$ and $K = \ker e_2^* = \operatorname{im}\omega$. Then, we have from (11.3)

$$J = \operatorname{im} e_2^*, \quad H^*(G/L_2^0) = J \oplus K, \quad \mathfrak{P}(J) = \mathfrak{P}(X_2).$$

Hence

(11.6) $$\mathfrak{P}(G/L^0) = \mathfrak{P}(X_2) + t^q\mathfrak{P}(\operatorname{im}\partial), \qquad \mathfrak{P}(G/L_2^0) = \mathfrak{P}(\operatorname{im}\partial) + \mathfrak{P}(K)/t^{q+1},$$
$$\mathfrak{P}(G/L_2^0) = \mathfrak{P}(K) + \mathfrak{P}(X_2).$$

Case I. Suppose X_2 to be orientable. Then from (3.5) and (11.2), we have $\mathfrak{P}(G/L^0) = \mathfrak{P}(X) = \mathfrak{P}(X_2)(1 + t^q)$. It follows then $\mathfrak{P}(X_2) = \mathfrak{P}(\operatorname{im}\partial)$,

and then $\operatorname{im}\omega = 0$, $\mathfrak{P}(G/L_2^0) = \mathfrak{P}(X_2)$. Since $\theta_2^*: H^*(G/L_2) \to H^*(G/L_2^0)$ is always injective, the equality $\mathfrak{P}(G/L_2^0) = \mathfrak{P}(X_2)$ tells us that θ_2^* is bijective. The first part of (11.5) is thus proved.

Case II. Suppose that X_2 is non-orientable. Let Ω be the characteristic class of the S^q-bundle $\{G/L^0, G/L_2^0, e_2\}$. For each l of L, we define $\phi(l)$: $G/L^0 \to G/L^0$ to be the homeomorphism $gL^0 \to glL^0$ and $\phi_2(l): G/L_2^0 \to G/L_2^0$ to be the homeomorphism $gL_2^0 \to glL_2^0$, $(g \in G)$. Then the pair $\{\phi(l), \phi_2(l)\}$ forms an automorphism of the bundle $\{G/L^0, G/L_2^0, e_2\}$. Therefore $\phi_2(l)^*\Omega = \pm\Omega$, and then Ω^2 is invariant under $\phi_2(l)^*$ for all l of L. Since $L_2 = LL_2^0$, Ω^2 is also left invariant by $\phi_2(L_2)$. The space G/L_2^0 is a covering space of G/L_2 with $\phi_2(L_2)$ as its group of covering transformations. Therefore $\Omega^2 \in \theta_2^*(H^*(G/L_2)) = J$. On the other hand, $e_2^*(\Omega^2) = 0$. (11.3) then tells us that $\Omega^2 = 0$.

We know that the linear map $\omega: H^*(G/L_2^0) \to H^*(G/L_2^0)$ is defined by $\omega(u) = \Omega \cdot u$, $u \in H^*(G/L_2^0)$. It follows that

$$\omega(K) = \omega(\operatorname{im}\omega) = 0, \qquad K = \omega(H^*(G/L_2^0)) = \omega(J + K) = \omega(J).$$

Here G/L_2^0 is an orientable manifold which covers the non-orientable manifold $X_2 = G/L_2$. Therefore [6] $\dim H^*(G/L_2^0) \geqq 2 \dim H^*(G/L_2) = 2 \dim J$. This together with (11.6) tells us that $\dim K \geqq \dim J$. But ω maps J onto K and so ω restricted to J is injective, whence $\mathfrak{P}(K) = t^{q+1}\mathfrak{P}(J) = t^{q+1}\mathfrak{P}(X_2)$. (11.6) then implies that

$$\mathfrak{P}(G/L_2^0) = (1 + t^{q+1})\mathfrak{P}(X_2), \qquad \mathfrak{P}(G/L^0) = (1 + t^{2q+1})\mathfrak{P}(X_2).$$

This together with (11.2) proves our proposition.

(11.7) *If $p = 1$ and q is even, then either* (a) $X_1 = S^q$, $X_2 = S^1$, $X = S^q \times S^1$, *or* (b) $n = 2q + 3$, $L_2^0 \neq 0$ *in* G, $\mathfrak{P}(X_1) = (1 + t^q)(1 + t^{q+1})$, $\mathfrak{P}(X_2) = (1 + t)(1 + t^{q+1})$, $X = S^1 \times X_1$ *and* X_2 *is orientable.*

Proof. Let us consider the restriction map $r: I(L_2^0) \to I(L^0)$. For each l of L, $\operatorname{Ad} l$ acts on both $I(L^0)$ and $I(L_2^0)$, and commutes with the restriction map r. Since L_2^0/L^0 is an even sphere, r is injective. From (11.1), $\operatorname{Ad} l$ acts trivially on $I(L^0)$, so it must act also trivially on $I(L_2^0)$ for all $l \in L$. But $L_2 = LL_2^0$, and hence $\operatorname{Ad} L_2$ leaves $I(L_2^0)$ elementwise fixed. Just as in the proof of (11.2), we know that the homomorphism $\theta_2^*: H^*(G/L_2) \to H^*(G/L_2^0)$ is bijective. X_2 is therefore orientable.

X is a q-sphere bundle over X_2 with $q > 1$. Therefore the homomorphism $f_2^\#: \pi_1(X) \to \pi_1(X_2)$ is bijective. (4.3) then implies that $\pi_1(X_1) = 0$,

whence L_1 is connected and X_1 orientable. On account of (3.5), the circle bundle $\{X, X_1, f_1\}$ is trivial. In particular, $X = S^1 \times X_1$.

Now consider the system $\{G, L^0, L_1, L_2{}^0\}$ of connected groups with $L^0 \subset L_1$, $L^0 \subset L_2{}^0$, $L_1 \subset G$, $L_2{}^0 \subset G$. Since both θ^* and $\theta_2{}^*$ are bijective, we have from (3.4) that

$$f_1{}^*(H^s(G/L_1)) + e_2{}^*(H^s(G/L_2{}^0)) = H^s(G/L^0),$$
$$f_1{}^*(H^s(G/L_1)) \cap e_2{}^*(H^s(G/L_2{}^0)) = 0 \qquad (0 < s < n-1).$$

Moreover, $L_1/L^0 = S^1$, $L^0/L_2{}^0 = S^q$. Using exactly the same argument as in the proof of (8.3), we find that either (i) $G/L_i = S^q$, $G/L_2{}^0 = S^1$, or (ii) $n = 2q+3$, $L_2{}^0 \not\sim 0$ in G, and

$$\mathfrak{P}(G/L_1) = (1+t^q)(1+t^{q+1}), \qquad \mathfrak{P}(G/L_2{}^0) = (1+t)(1+t^{q+1}).$$

These together with (11.6) and the orientability of X_2 prove our proposition.

(11.8) *If $p = 1$ and q is odd, then either* (a) *X_1, X_2 are points, or* (b) *X_1, X_2 have the same real homology as some odd-dimensional spheres.*

Proof. Let us consider the restriction map $r_i: I(L_i{}^0) \to I(L^0)$, $(i = 1, 2)$, Since $L_i{}^0/L^0$ is an odd sphere, r_i is surjective. For any l of L, $\mathrm{Ad}\, l$ acts on both $I(L^0)$ and $I(L_i{}^0)$, and commutes with r_i. By (11.1), $I(L^0)$ is elementwise invariant under $\mathrm{Ad}\, L$. Since L is compact, to each $u \in I(L^0)$, there exists $v_i \in I(L_i{}^0)$ such that $r_i(v_i) = u$ and v_i is invariant under $\mathrm{Ad}\, L$. But $L_i = LL_i{}^0$ and so v_i is also $\mathrm{Ad}\, L_i$-invariant. This tells us that v_i belongs to $I(L_i)$. Hence the restriction map $\alpha_i: I(L_i) \to I(L)$ is also surjective. It follows then from (3.3) and the following commutative diagram

$$\begin{array}{ccc} I(L_1) & \xrightarrow{\sigma_1} & H^*(X_1) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle f_1{}^*} \\ I(L) & \xrightarrow{\sigma} & H^*(X) \\ \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle f_2{}^*} \\ I(L_2) & \xrightarrow{\sigma_2} & H^*(X_2) \end{array}$$

that $\sigma(I(L))$ contains only element of zero degree. But we know that $I(L) = I(L^0)$ and $\theta^*: H^*(G/L) \to H^*(G/L^0)$ is bijective. Therefore the characteristic subalgebra of the homogeneous space G/L^0 is trivial and then the projection $H^*(G/L^0) \to H^*(G)$ is injective. Since we know that θ^*, $f_i{}^*$ are all injective, the homomorphism $h^*: H^*(X) \to H^*(G)$, $h_i{}^*: H^*(X_i)$

$\to H^*(G)$ induced by the projections are all injective $(i=1,2)$. Taking account of (11.5) and using the same argument as used in the proof of (5.2), we find that either $\mathfrak{P}(X_1)=\mathfrak{P}(X_2)=1$, or $\mathfrak{P}(X_1)=1+t^a$, $\mathfrak{P}(X_2)=1+t^b$, where a, b are two odd numbers. (11.8) is thus proved.

(11.9) *Suppose that* $p=1$ *and* q *is odd. Then either* (i) X_1, X_2 *are points,* (ii) $X_1=S^q$, $X_2=S^1$, $X=S^q\times S^1$, (iii) $n=2q+3$, $\mathfrak{P}(X)=(1+t)(1+t^{2q+1})$, $\mathfrak{P}(X_1)=1+t^{2q+1}$, $\mathfrak{P}(X_2)=1+t$, X_2 *is non-orientable,* (iv) $n=5$, $p=q=1$,

$$\mathfrak{P}(X_1)=1+t,\quad \mathfrak{P}(X_2)=1+t^3,\quad \mathfrak{P}(X)=(1+t)(1+t^3),$$

(v) $n=4$, $p=q=1$, $\mathfrak{P}(X_1)=\mathfrak{P}(X_2)=1$, $\mathfrak{P}(X)=1+t^3$, *both* X_1 *and* X_2 *are non-orientable, or* (vi) $n=7$, $p=q=1$, $\mathfrak{P}(X)=(1+t^3)^2$, $\mathfrak{P}(X_1)=\mathfrak{P}(X_2)=(1+t^3)$, *both* X_1 *and* X_2 *are non-orientable.*

Proof. Let us divide our discussions into four cases according to the orientability of X_1 and X_2.

Case a. Both X_1 and X_2 are orientable. From (11.5) and (11.8), we know that either X_1 and X_2 are points, or X_1 and X_2 are homology spheres with real coefficients of odd dimension. In the second alternative, it follows from (3.4) and the orientability of X_1, X_2 that $\dim X_1+\dim X_2=\dim X$, and then (4.8) tells us that $X_1=S^q$, $X_2=S^1$, $X=S^q\times S^1$. Thus (11.9) is proved for this case.

Case b. X_1 is orientable but X_2 is not. From (3.4), (3.5) and (11.5), we have

$$\mathfrak{P}(X)=(1+t)\mathfrak{P}(X_1)=(1+t^{2q+1})\mathfrak{P}(X_2)=\mathfrak{P}(X_1)+\mathfrak{P}(X_2)+t^{n-1}-1,$$

whence

$$\mathfrak{P}(X_1)=(1+t^{2q+1})(1-t^{n-1})/(1-t^{2q+2}),$$

$$\mathfrak{P}(X_2)=(1+t)(1-t^{n-1})/(1-t^{2q+2}).$$

By (11.8), X_1 and X_2 are homology spheres, and therefore,

$$n=2q+3,\ \mathfrak{P}(X_1)=1+t^{2q+1},\ \mathfrak{P}(X_2)=1+t.\ \mathfrak{P}(X)=(1+t)(1+t^{2q+1}).$$

(11.9) is hence valid in this case.

Case c. X_2 is orientable but X_1 is not. Suppose $q>1$. Then $f_2^\#$: $\pi_1(X)\to\pi_1(X_2)$ is bijective and then, by (4.3), X_1 is simply-connected. This contradicts the non-orientability of X_1, and therefore, we must have $q=1$.

Thus this case is reduced to a particular case of Case b with X_1 and X_2 interchanged. We have then

$$\mathfrak{P}(X_1) = 1 + t,\ \mathfrak{P}(X_2) = 1 + t^3,\ \mathfrak{P}(X) = (1 + t)(1 + t^3),\ n = 5.$$

Case d. Both X_1 and X_2 are non-orientable. From (4.3), we must have $q = 1$, and then, from (11.5) and (3.4)

$$\mathfrak{P}(X) = (1 + t^3)\mathfrak{P}(X_1) = (1 + t^3)\mathfrak{P}(X_2) = \mathfrak{P}(X_1) + \mathfrak{P}(X_2) + t^{n-1} - 1,$$

whence

$$\mathfrak{P}(X_1) = \mathfrak{P}(X_2) = (1 - t^{n-1})/(1 - t^3).$$

But from (11.8), X_1 is a homology sphere and therefore either $n = 4$, $\mathfrak{P}(X) = 1 + t^3$, $\mathfrak{P}(X_1) = \mathfrak{P}(X_2) = 1$, or $n = 7$, $\mathfrak{P}(X) = (1 + t^3)^2$, $\mathfrak{P}(X_1) = \mathfrak{P}(X_2) = 1 + t^3$. This completes the proof.

12. Determination of G when $p = 1$. Throughout this section, we always assume that $p = 1$. Let us first study the case by which q is even. Then L_1 is connected, X_1 simply-connected and $X = S^1 \times X_1$. Neglecting the case $X_1 = S^q$, $X_2 = S^1$, we suppose that Case b in (11.7) happens. Then $L_2^0 \neq 0$ in G and

$$n = 2q + 3,\ \mathfrak{P}(X_1) = (1 + t^q)(1 + t^{q+1}),\ \mathfrak{P}(X_2) = (1 + t)(1 + t^{q+1}).$$

Using the same kind of arguments as (but much simpler than) those used in §§8, 9, 10, we find that, when $n > 15$, then up to a finite covering,

$$G = O(2) \times O(q + 2),\ L^0 = O(q),\ L_1 = L_1^0 = T \cdot O(q),\ L_2^0 = O(q + 1), \tag{12.1}$$
$$T \not\subset O(2),\ T \cap O(q + 2) = e,$$

where T is a circle group centralizing L^0, $O(r)$ denotes the rotation group in r variables, and the imbeddings $O(q) \subset O(q + 1)$, $O(q + 1) \subset O(q + 2)$ are in the canonical manner. Here, unlike the situation in § 10, the subgroups L, L_1, L_2 are not uniquely determined by these properties. We have to discuss the components of L and the position of T in G. For simplicity, let us assume

$$O(2) \cap L = e. \tag{12.2}$$

No generality is lost here since $O(2) \cap L$ leaves every point of S^n fixed. We note that after factoring out $O(2) \cap L$ from $O(2)$, G remains the same form as direct product.

(12.3) *Under the above assumptions, L and L_2 have at most two connected components.*

Proof. Since L_2/L is an even sphere, L_2 and L have the same number of connected components, and hence it is sufficient for us to discuss L_2/L_2^0. Let $\xi: G \to O(q+2)$ be the projection. Then $\xi(L_2)$ belongs to the normalizer of L_2^0 in $O(q+2)$. But from (12.1), L_2^0 is an $O(q+1)$ imbedded in $O(q+2)$ in the canonical manner, and so $\xi(L_2)/L_2^0$ is at most of order 2. Now let us consider the intersection $O(2) \cap L_2$ of L_2 and the kernel of ξ. Since $L \subset L_1$, there exist elements $a_1, a_2, \cdots, a_s \in T$ such that $L = \bigcup_i a_i L^0$, whence $L_2 = \bigcup_i a_i L_2^0$. Suppose $b \in O(2) \cap L_2$. There is an element c of L_2^0 such that $b = a_i c$ for a certain i. Both b and a_i centralize L^0, so does c. Since q is even, the centralizer of $O(q)$ in $O(q+1)$ belongs to $O(q)$. Therefore, $c \in O(q) = L^0$ and then $b \in L$. On account of (12.2), b must be the identity. In other words, $O(2) \cap L_2 = e$, whence $L_2/L_2^0 \approx \xi(L_2)/L_2^0$ is at most of order 2. (12.3) is thus proved.

(12.4) *The assumptions are as above. Let m be the order of the cyclic group $O(2) \cap T$, $e_1: G/L^0 \to G/L_1$ the natural map, and Y the submanifold $O(2) \times O(q+1)/O(q)$ of G/L^0. Denote by $[u]$ the integral homology class of the fundamental cycle of Y. Then e_1 carries $[u]$ into $2m$ times a generator of the $(q+1)$-th homology group $H_{q+1}(G/L_1, Z)$ with integer coefficients.*

Proof. Let C be the identity component of the centralizer of L^0 in $O(q+2)$. From (12.1), C is a circle group. Since $T \cap O(q+2) = e$, and $O(2) \cap T$ is of order m, we can write

$$T = \{(a, \beta(a)^m) : a \in O(2)\},$$

where $\beta :: O(2) \to C$ is one of the two isomorphisms of $O(2)$ on C. Now let us consider the following two maps

$$G/L_1 \to O(q+2)/O(q), \qquad O(q+2)/O(q) \to O(q+2)/O(q+1)$$

defined, respectively, by

$$agL_1 \to g\beta(a)^{-m}O(q), \qquad gO(q) \to gO(q+1),$$
$$a \in O(2), \qquad g \in O(q+2).$$

The first map is bijective while the second is the projection in the tangent bundle over an S^{q+1}. Since $q+1$ is odd, the composite map

$$j: G/L_1 \to O(q+2)/O(q+1)$$

of the above two maps induces an isomorphism between the $(q+1)$-th integral homology groups $H_{q+1}(G/L_1^0)$ and $H_{q+1}(O(q+2)/O(q+1))$. Thus, in order

to prove our proposition, it suffices to prove that the map

$$j \circ e_i : Y \to O(q+2)/O(q+1)$$

is of degree $2m$. For this purpose, let us note that $O(q+2)/O(q+1) = S^{q+1}$, $Y = O(2) \times O(q+1)/O(q) = S^1 \times S^q$, and the map $j \circ e_1$ is given by $agO(q) \to g\beta(a)^{-m}O(q+1)$, $a \in O(2)$, $g \in O(q+1)$. By using orthogonal matrices, we can, in a very natural manner, identify Y with the subset

$$\{(\cos\theta, -\sin\theta, x_1, x_2, \cdots, x_{q+1}) : 0 \leqq \theta \leqq 2\pi, \sum x_i^2 = 1\}$$

of R^{q+3} and identify S^{q+1} with the subset $\{(y_0, y_1, \cdots, y_{q+1}) : \sum y_i^2 = 1\}$ of R^{q+2} such that $j \circ e_1$ is given by

$$y_0 = \cos m\theta, \quad y_i = -x_i \sin m\theta, \qquad i = 1, 2, \cdots, q+1.$$

It is direct to verify that $j \circ e_1$ is of degree $2m$ either by local degree argument or by using differential forms. (12.4) is thus proved.

(12.5) *The assumptions are as above. Then $O(2) \cap T = e$, and L, L_2 have exactly two connected components.*

Proof. Let Z_k denote the additive group of integers mod k, $k \neq 1$. Then the commutative diagram

$$\begin{array}{ccccc} & & G/L^0 & \xrightarrow{e_2} & G/L_2^0 \\ & \swarrow^{e_1} & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_2} \\ X_1 & & & & \\ & \nwarrow_{f_1} & X & \xrightarrow{f_2} & X_2 \end{array}$$

induces a commutative diagram of the $(q+1)$-th homology groups with Z_k as the coefficient group

$$\begin{array}{ccccc} & & H_{q+1}(G/L^0, Z_k) & \xrightarrow{e_2} & H_{q+1}(G/L_2^0, Z_k) \\ & \swarrow^{e_1} & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_2} \\ H_{q+1}(X_1, Z_k) & & & & \\ & \nwarrow_{f_1} & H_{q+1}(X, Z_k) & \xrightarrow{f_2} & H_{q+1}(X_2, Z_k). \end{array}$$

$Y = O(2) \times O(q+1)/O(q)$ is a submanifold of $G/L^0 = O(2) \times O(q+2)/O(q)$. Let u be the fundamental cycle of Y, and $[u]_k$ the element of $H_{q+1}(G/L^0, Z_k)$ which contains u. Since $q+1$ is odd, we know from the familiar properties

of the Stiefel manifolds that $[u]_k \neq 0$ for any $k \neq 1$. The map e_2 carries Y into a circle, and hence

$$e_2([u]_k) = 0, \qquad f_2\theta([u]_k) = \theta_2 e_2([u]_k) = 0.$$

From (12.4), $e_1([u]_{2m}) = 0$, and then by the above commutative diagram, $f_1\theta([u]_{2m}) = 0$. Therefore $\theta([u]_{2m})$ belongs to both the kernel of f_1 and the kernel of f_2. Since $0 < q+1 < n-1$, it follows from the dual of (3.4) that $\theta([u]_{2m}) = 0$.

If L is connected, then θ becomes the identity map, and then $[u]_{2m} = \theta([u]_{2m}) = 0$ which is impossible. Therefore by (12.3), both L and L_2 have exactly two connected components. Now we know that $\theta: G/L^0 \to X$ is a two-fold covering map, and moreover, from (11.2) that θ induces an isomorphism on the homology groups with *real* coefficients. Therefore, $\theta([u]_k) \neq 0$ for $k > 2$. This implies that $2m \leqq 2$, or in other words, $m = 1$, $O(2) \cap T = e$. (12.5) is thus proved.

(12.6) *The assumptions are as above. Then* $(O(2) \times O(q+2))/I_2$ *acts effectively on* S^n, *where* I_2 *is the only central subgroup of order 2 which belongs to neither* $O(2)$ *nor* $O(q+2)$. *Moreover, the triple* $\{L, L_1, L_2\}$ *of subgroups is uniquely determined up to an automorphism of* G.

Proof. From (12.1), the pair $\{L^0, L_2{}^0\}$ of subgroups of $G = O(2) \times O(q+2)$ is unique up to an automorphism of $O(q+2)$. From (12.1) and (12.5), T is a circle group centralizing L^0 and $T \cap O(2) = T \cap O(q+2) = e$. Therefore T is unique up to an automorphism of $O(2)$. Now let us see that T, L^0, $L_2{}^0$ completely determine the triple $\{L, L_1, L_2\}$. Since $L_1 = T \cdot L^0$, L_1 is settled. Noting that L/L^0 is of order 2, $L \subset L_1$ and $L \cap O(2) = e$, we find immediately that L can only be $L_2 L$. Therefore, we have

$$L_1 = T \cdot L^0, \; L = I_2 L^0, \; L_2 = L L_2{}^0 = I_2 L_2{}^0.$$

Thus, up to an automorphism of $O(2) \cap O(q+2)$, the system $\{T, L^0, L_2{}^0\}$ and hence the system $\{L, L_1, L_2\}$ is unique. This establishes the second part of (12.6). The first part of (12.6) follows from the fact that I_2 is the only normal subgroup of G contained in L. The proof of our proposition is completed.

As a direct consequence of (11.5), (12.1) and (12.6), we have

(12.7) *Let* G *be a compact, connected differentiable transformation group of* S^n *with* $(n-1)$*-dimensional orbits. Suppose that* $n > 15$, $p = 1$, q *is even, and the special orbits are not spheres. Then* $n = 2q+3$, X_1 *is the Stiefel manifold* $V_{q+2,2}$, $X = S^1 \times X_1$, *and* X_2 *is an orientable* S^{q+1}*-bundle*

over S^1. Moreover, if G is effective, then $G \approx (O(2)\times O(q+2))/I_2$, where I_2 is the central subgroup of $O(2)\times O(q+2)$ belonging to neither of the direct factors. The system $\{L, L_1, L_2\}$ of subgroups is unique up to an automorphism of G.

Now let us come to the case by which $p=1$ and q is odd. Taking account of (11.9) and using similar arguments as in the case of even q, we can prove

(12.8) *Let G be a compact, connected differentiable transformation group of S^n with an $(n-1)$-dimensional orbit. Suppose that $p=1$, q is odd, $n>13$, and the special orbits are neither both points nor (S^q, S^1). Then $n=2q+3$, X_1 is the Stiefel manifold $V_{2q+2,2}$, $X=S^1\times X_1$, and X_2 is the $(q+2)$-dimensional generalized Klein bottle. If G is, moreover, effective, then $G\approx O(2)\times O(q+2)$ and the triple $\{L, L_1, L_2\}$ of subgroups is unique up to an automorphism of G.*

13. Summary of the results. For clarity, let us recall some of the relevant definitions and summarize the results which have been obtained. G is a compact, connected and differentiable transformation group of S^n with an $(n-1)$-dimensional orbit. X denotes a general orbit, X_1 and X_2 denote the two special orbits, and $p=\dim X-\dim X_1$, $q=\dim X-\dim X_2$. L, L_1, L_2 are the isotropic subgroups of G at points a, a_1, a_2 on X, X_1, X_2 respectively, where a, a_1, a_2 are so chosen that $L\subset L_1$, $L\subset L_2$. From (5.2), (7.4), (8.3), (10.1), (10.4), (11.7), (11.9), (12.7) and (12.8), we have

THEOREM I. *If n is even and different from 4, then either* (i) *X_1, X_2 are single points, $X=S^{n-1}$, or* (ii) *$X_1=S^q$, $X_2=S^p$, $X=S^q\times S^p$.*

THEOREM II. *Suppose that G is effective, $n>31$ is odd, and $q\leqq p$ (up to a change of notation, we can always assume $q\leqq p$). Then there are only five possibilities:*

(i) *X_1, X_2 are single points, $X=S^{n-1}$;*

(ii) *$X_1=S^q$, $X_2=S^p$, $X=S^q\times S^p$;*

(iii) *$q=4$, $p=4r-5$, $n-1=2(p+4)$, X_1 is an S^{p+4}-bundle over S^4, $X=S^4\times X_2$, X_2 is the quaternion Stiefel manifold $Sp(r)/Sp(r-2)$, $G\approx (Sp(r)\times Sp(2))/I_2$, where I_2 is the central subgroup of order 2 belonging to neither $Sp(r)$ nor $Sp(2)$;*

(iv) *$q=2$, $p=2r-3$, $n-1=2(p+2)$, X_1 is an S^{p+2}-bundle over S^2, $X=S^2\times X_2$, X_2 is the complex Stiefel manifold $SU(r)/SU(r-2)$, and G is isomorphic with either $(SU(r)\times SU(2))/I$ or $(U(r)\times U(2))/T$, where*

I and T are the maximal central subgroups which intersect each direct factor only at the indentity;

(v) *$q=1$, $n=2p+3$, X_1 is an S^{p+1}-bundle over S^1, X_2 is the real Stiefel manifold $O(p+2)/O(p)$, $X=S^1\times X_2$, and G is isomorphic with $O(p+2)\times O(2)$ or $(O(p+2)\times O(2))/I_2$ according as p is odd or even, where I_2 denotes the central subgroup of order 2 belonging to neither of the direct factors. Moreover, in Cases* (iii), (iv) *and* (v), *the triple $\{L, L_1, L_2\}$ of subgroups is unique up to an automorphism of G.*

Concerning the orbits of G, we know from the above theorems that there are five possibilities. Now let us see that all of them are actually realized by subgroups of $O(n+1)$ acting on the unit sphere in R^{n+1}.

(i) Let G be any closed and connected subgroup of $O(n)$ transitive on the unit sphere S^{n-1} of R^n. By imbedding $O(n)$ in $O(n+1)$ in the canonical manner, the group G acts on the unit sphere S^n of R^{n+1} with an $(n-1)$-dimensional orbit and with two single points as the special orbits.

(ii) Let G', G'' be connected closed subgroups of $O(q+1)$, $O(p+1)$ transitive on the unit spheres S^q, S^p of R^{q+1}, R^{p+1} respectively. Since $p+q=n-1$, we can identify the direct sum $R^{q+1}\otimes R^{p+1}$ with R^{n+1}, and thus G', G'' are imbedded in $O(n+1)$ as subgroups. The direct product $G'\times G''$ acts on the unit sphere of R^{n+1} with an $(n-1)$-dimensional orbit and with S^q, S^p as the two special orbits.

(iii) Let $Sp(r)$ be the group of all symplectic transformations of the r-dimensional right quaternion vector space Q^r and $Sp(2)$ that of the left quaternion plane Q^2. Then $Sp(r)\times Sp(2)$ acts naturally (not effectively) on the tensor product $Q^r\otimes Q^2$ which is the real vector space R^{8r}. Let $Sp(r)\otimes Sp(2)$ denote the group of automorphisms of R^{8r} so obtained. Then $Sp(r)\otimes Sp(2)$ acts on the unit sphere S^n of R^{8r}, $n=8r-1$, with an $(n-1)$-dimensional orbit. The special orbits consist of a quaternion Stiefel manifold and an S^{4r-1}-bundle over S^4. We note that $Sp(r)\times Sp(2)$ is a two-fold covering group of $Sp(r)\otimes Sp(2)$.

(iv) The tensor products $SU(r)\otimes SU(2)$ and $U(r)\otimes U(2)$ are groups of unitary transformations of the complex vector space C^{2r}, and hence are transformation groups of the unit sphere S^n of C^{2r}, where $n=4r-1$. Both of these groups have an $(n-1)$-dimensional orbit. The two special orbits consists of a complex Stiefel manifold and an S^{p+2}-bundle over S^2. It is noted that $SU(r)\otimes SU(2)\approx (SU(r)\times SU(2))/I$, $U(r)\otimes U(2)\approx (U(r)\times U(2))/T$,

where I is the identity for odd r and is of order 2 for even r, and T is a circle group.

(v) The tensor product $G = O(p+2) \otimes O(2)$ which acts on R^{2p+4} orthogonally gives a realization of the possibility (v) in Theorem II.

14. Orthogonality of G. Let K be a transformation group of a space M. By a linear representation of $\{K, M\}$ of degree m, we mean a pair $\{\alpha, \phi\}$, where ϕ is a continuous map of M into the euclidean space R^m and $\alpha: K \to GL(R, m)$ is a homomorphism such that

$$\alpha(k)(\phi(x)) = \phi(k(x)),\ k \in K,\ x \in M.$$

Now suppose that M is an n-sphere. We say that K acts on M orthogonally if there exists a linear representation $\{\alpha, \phi\}$ of degree $n+1$ such that ϕ carries M homeomorphically onto the unit sphere in R^{n+1}. It is to be noted that we do not assume K to be effective and hence α may not be an isomorphism.

It is the aim of the present section to show that if $n > 31$, then G acts on S^n orthogonally. For this purpose, we find it convenient to write the difference set $S^n - (X_1 \cup X_2)$ in the form of a topological product. Let us recall that, in our discussions, we take a cross section $\overset{\frown}{a_1 a_2}$ of the orbits of G. This cross section is an arc with a_1, a_2 as the end points which has the properties that the isotropc subgroups at any two inner points coincide, and $G(a_1) = X_1$, $G(a) = X_2$ are the two special orbits. Let us parametrize this arc by t in such

$$\overset{\frown}{a_1 a_2} = \{s(t) : -1 \leqq t \leqq 1\},\ s(-1) = a_1,\ s(1) = a_2,$$

and $G(s(0))$ is our orbit X. This point $s(0)$ coincides with the point "a" which we used in § 3. There is a natural homeomorphism between the product $I_0 \times X$ and the difference set $S^n - (X_1 \cup X_2)$, where $I_0 = \{t : -1 < t < 1\}$. In fact, for any (t, x) with $x \in X$, $-1 < t < 1$, there exists $g \in G$ such that $g(s(0)) = x$. Since all the isotropic subgroups at $s(t)$ coincide when $-1 < t < 1$, the point $g(s(t))$ is independent of the choice of g. Thus we get a map $(t, x) \to g(a(t))$ which can be easily verified to be continuous and bijective. From now on, we identify $I_0 \times X$ with $S^n - (X_1 \cup X_2)$ by means of this map. It is evident that

$$g(t, x) = (t, g(x)),\ -1 < t < 1,\ g \in G,\ x \in X.$$

THEOREM III. *Let G be a compact, connected differentiable transformation group of S^n with an $(n-1)$-dimensional orbit. If the special orbits are either* (i) *both single points, or* (ii) *spheres S^q, S^p with $p + q = n - 1$, then G acts on S^n orthogonally.*

Proof. Case (i) has been proved by Poncet [17]. It suffices for us to study the case by which $X_1 = S^q$, $X_2 = S^p$ with $p+q = n-1$. Since G is compact and connected and acts on X_i transitively, it must act on X_i orthogonally $(i = 1, 2)$ [17]. There exist then a homomorphism α_1: $G \to GL(R, q+1)$ and a map ϕ_1: $X_1 \to R^{q+1}$ such that $\alpha_1(g)\phi_1(x_1) = \phi_1(g(x_1))$, $x_i \in X_i$, $g \in G$, and ϕ_1 carries X_1 homeomorphically onto the unit sphere in R^{q+1}. Similarly, we have a homomorphism α_2: $G \to GL(R, p+1)$ and a map ϕ_2: $X_2 \to R^{p+1}$ such that $\alpha_2(g)\phi_2(x_2) = \phi_2(g(x_2))$, $x_2 \in X_2$, $g \in G$ and ϕ_2 carries X_2 homeomorphically onto the unit sphere of R^{p+1}.

Let us define ϕ: $S^n \to R^{q+1} \oplus R^{p+1} = R^{n+1}$ as follows:

$$\begin{cases} \phi(x_1) = (\phi_1(x_1), 0), \ x_1 \in X_1, \\ \phi(x_2) = (0, \phi_2(x_2)), \ x_2 \in X_2. \\ \phi(t, x) = ((t+1)\phi_1 f_1(x), (t-1)\phi_2 f_2(x))/[(t-1)^2 + (t+1)^2]^{\frac{1}{2}}, \\ \qquad\qquad (t, x) \in I_0 \times X = S^n - (X_1 \cup X_2). \end{cases}$$

Here, $\phi_1 f_1(x)$ and $\phi_2 f_2(x)$ are points in euclidean spaces, and so the multiplication by real numbers has a meaning. It is evident that ϕ is continuous and $\phi(S^n)$ belongs to the unit sphere of R^{n+1}. From (4.8), ϕ is injective and hence ϕ carries S^n homeomorphically onto the unit sphere of R^{n+1}. Let us consider R^{q+1}, R^{p+1} to be subspaces of R^{n+1} in the natural manners. Thus $GL(R, q+1)$, $GL(R, p+1)$ are imbedded in $GL(R, n+1)$ as subgroups. Defining α: $G \to GL(R, n+1)$ by $\alpha(g) = \alpha_1(g)\alpha_2(g)$, and taking account of the facts that, for $i = 1, 2$.

$$\alpha_i(g)(\phi_i(f_i(x))) = \phi_i(g(f_i(x))), f_i(g(x)) = g(f_i(x)), \ g \in G, \ x \in X,$$

we find immediately that the pair $\{\alpha, \phi\}$ forms a linear representation of $\{G, S^n\}$ of degree $n+1$. Hence G acts on S^n orthogonally. Theorem III is thus proved.

To discuss the orthogonality of G when the special orbits are neither points nor spheres, we need the following proposition:

(14.1) *Let G be a compact, connected differentiable group acting on S^n with an $(n-1)$-dimensional orbit, and L, L_1, L_2 have the same meaning as before. Suppose that G also acts on another sphere S^m with an $(n-1)$-dimensional orbit, and that the corresponding isotropic subgroups be L', L_1', L_2'. If there exists an automorphism θ of G carrying $\{L, L_1, L_2\}$ into $\{L', L_1', L_2'\}$, then $n = m$ and the two transformation groups $\{G, S^n\}$, $\{G, S^m\}$ are equivalent.*

Proof. Since $n-1=\dim G-\dim L=\dim G-\dim L'=m-1$, we have $n=m$. Let $\widehat{a_1a_2}$ ($\widehat{a_1'a_2'}$) be the cross section in S^n (S^m) of orbits based on which L, L_1, L_2 (L', L_1', L_2') are defined. We write

$$\widehat{a_1a_2}=\{s(t):-1\leqq t\leqq 1\},\qquad \widehat{a_1'a_2'}=\{s'(t):-1\leqq t\leqq\}$$

such that

$$s(-1)=a_1,\quad s(1)=a_2,\quad s'(-1)=a_1',\quad s'(1)=a_2'.$$

This facilitates us to construct a homeomorphism of S^n onto S^m. In fact, let $x\in S^n$. There exists $g\in G$ and t, $-1\leqq t\leqq 1$, such that $x=g(s(t))$. Here t is uniquely determined by x while g is not. Suppose that $x=h(s(t))$, $h\in G$; then $g^{-1}h$ belongs to L_1, L_2 or L according as $t=-1$, $t=1$ or $-1<t<1$. Let us define ϕ: $S^n\to S^m$ by $\phi(x)=\theta(g)(s'(t))$, where g and t satisfy $x=g(s(t))$. Since $\theta(L)=L'$, $\theta(L_1)=L_1'$, $\theta(L_2)=L_2'$, L_1' leaves a_1' invariant, L_2' leaves a_2' invariant and L' leaves $\widehat{a_1'a_2'}$ pointwise invariant, it follows that ϕ is independent of the choice of g and hence well-defined. We can easily verify that ϕ is continuous and injective. Therefore ϕ maps S^n homeomorphically onto S^m as they have the same dimension. Moreover, from the definition of ϕ, we have $\phi(g(x))=\theta(g)(\phi(x))$, $x\in S^n$, $g\in G$. The two transformation groups $\{G, S^n\}$ and $\{G, S^m\}$ are equivalent.

THEOREM IV. *Suppose that n is either even and different from* 4, *or odd and greater than* 31. *Then any compact and connected differentiable transformation group of S^n with an $(n-1)$-dimensional orbit is orthogonal.*

Proof. If the special orbits are both points or both spheres, then this is contained in Theorem III. Therefore, it is sufficient to discuss the Cases (iii), (iv), and (v) in Theorem II. From (14.1), the transformation group $\{G, S^n\}$ is determined by the system $\{G, L, L_1, L_2\}$, and from Theorem II, the system $\{G, L, L_1, L_2\}$ is completely determined by the topological group G. Hence, for a given n, there is at most one transformation group G (up to equivalence) realizing Case (iii) or Case (v), while there are at most two transformation groups realizing Case (iv). It follows that the orthogonal models which we have constructed in § 13 exhaust all the transformation groups G for Cases (iii), (iv) and (v). Theorem IV is thus proved.

15. Some exceptional cases. In § 7, we have classified all the orbits of G when both p and q are even. By using (14.1), the transformation groups $\{G, S^n\}$, in this case, can be quite easily determined. We shall give these

groups explicitly in order to see some examples different from the five main classes mentioned in Theorem IV.

THEOREM V. *Let G be a compact, connected differentiable transformation group of S^n with an $(n-1)$-dimensional orbit such that the differences of the dimensions of the orbits are always even. Then n is odd and G acts on S^n orthogonally. If the special orbits are neither both single points nor both spheres, then the transformation group G is either the linear adjoint group of a simple group of rank 2, or the irreducible representation of C_3 of degree 14, or the irreducible representation of the group F_4 of degree 26.*

Proof. The fact that n is odd follows directly from (7.4). On account of Theorem I, we can assume that the special orbits are neither both points nor both spheres. Therefore, by (7.4), there are only five possibilities: (a) $G=A_2$, $L=T^2$, L_1 and L_2 are locally isomorphic with $T\times A_1$; (b) $G=C_2$, $L=T^2$, L_1 and L_2 are locally isomorphic with $T\times A_1$, (c) $G=G_2$, $L=T^2$, L_1 and L_2 are locally isomorphic with $T\times A_1$, (d) $G=C_3$, L is locally isomorphic with $C_1\times C_1\times C_1$, L_1 and L_2 are locally isomorphic with $C_1\times C_2$, (e) $G=F_4$, $L=D_4$, L_1 and L_2 are of Cartan class B_4. If G is, moreover, effective, than G has trivial center. We can verify directly that the compact linear groups given in our theorem give orthogonal realizations of these five cases. Hence, to prove our theorem, it suffices to prove that, for a given G, the action of G of S^n is unique. On account of (14.1), it is sufficient to show that, for each G, the triple $\{L, L_1, L_2\}$ of subgroups is unique up to an automorphism of G. Let us discuss the five cases separtely. We shall give the detail for $G=C_3$. As for the rest, only indications will be provided.

1. $G=C_3$. Then L is locally isomorphic with $C_1\times C_1\times C_1$, and L_1, L_2 with $C_1\times C_2$. Let $\{L', L_1', L_2'\}$ be another such triple of subgroups of G. We shall show that there exists an automorphism of G carrying L, L_1, L_2 into L', L_1', L_2' respectively. Since both L, L' are connected and locally isomorphic with $C_1\times C_1\times C_1$, they must be conjugate [14], and thus we can assume $L=L'$. Take a maximal toral subgroup H of L and consider it Lie algebra $\boldsymbol{H}$. Evidently $\boldsymbol{H}$ is a Cartan subalgebra of the Lie algebra $\boldsymbol{G}$ of G. We can choose coordinates τ_1, τ_2, τ_3 of $\boldsymbol{H}$ such that the angular parameters are

$$\pm 2\tau_i, \pm\tau_i\pm\tau_j \qquad (i,j=1,2,3).$$

For each angular parameter ρ, let us denote by Y_ρ, Z_ρ the quasi-root vectors [14] corresponding to $\pm\rho$. It is known that any subalgebra containing $\boldsymbol{H}$ is spanned by $\boldsymbol{H}$ and some quasi-root vectors. We see immediately that the

Lie algebra $\boldsymbol{L}$ of L is spanned by $\boldsymbol{H}$ and $Y_{2\tau_i}$, $Z_{2\tau_i}$, $i=1,2,3$. A direct computation tells us that there are only three subalgebras $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{J}_3$ of $\boldsymbol{G}$ which contain $\boldsymbol{L}$ and are of type $C_1 \times C_2$, namely

$$\boldsymbol{J}_1 = \{\boldsymbol{L}, Y_{\tau_1+\tau_2}, Z_{\tau_1+\tau_2}, Y_{\tau_1-\tau_2}, Z_{\tau_1-\tau_2}\},$$
$$\boldsymbol{J}_2 = \{\boldsymbol{L}, Y_{\tau_1+\tau_3}, Z_{\tau_1+\tau_3}, Y_{\tau_1-\tau_3}, Z_{\tau_1-\tau_3}\},$$
$$\boldsymbol{J}_3 = \{\boldsymbol{L}, Y_{\tau_2+\tau_3}, Z_{\tau_2+\tau_3}, Y_{\tau_2-\tau_3}, Z_{\tau_2-\tau_3}\}.$$

We know that the Weyl group consists of transformations of the form

$$\tau_j = \epsilon_j \tau_{\sigma(j)}, \qquad \epsilon_j = \pm 1, \qquad\qquad j = 1,2,3,$$

where σ denotes a permutation of the indices. It follows that, given any two pairs of the algebras $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{J}_3$, there exists an automorphism of $\boldsymbol{G}$ leaving $\boldsymbol{L}$ invariant and carrying one pair to the other. In other words, there is essentially only one pair of subalgebras of the type $C_1 \times C_2$ which contains $\boldsymbol{L}$. Since L_1, L_2, L_1', L_2' are connected and $L_1 \neq L_2$, $L_1' \neq L_2'$ on account of (6.2), we can find an automorphism of G leaving L invariant and carrying L_1, L_2 into L_1', L_2' respectively. Our theorem is thus proved in this case.

2. $G = F_4$. The proof is the same as before. We note the following two properties of F_4 which can be readily verified: (1) up to an automorphism of F_4, there is only one connected subgroup of type D_4; (2) given any two pairs of connected subgroups of type B_4 which contains a fixed subgroup L of type D_4; there exists an inner automorphism of F_4 leaving L invariant and carrying one pair to the other.

3. $G = A_2$. The proof is the same as the case $G = C_3$.

4. $G = G_2$. L is a two-dimensional toral group and L_1, L_2 are locally isomorphic with $T \times A_1$. There are six connected subgroups which contain L and are locally isomorphic with $T \times A_1$, and hence 15 pairs of such subgroups. Up to automorphisms of G leaving L invariant, there are only four different pairs. Among these four pairs, only one pair satisfies both (6.2) and (7.5). Therefore, the triple $\{L, L_1, L_2\}$ of subgroups of G_2 is unique up to automorphisms of G_2.

5. $G = C_2$. The proof is the same as the preceding one.

REFERENCES.

[1] A. Borel, "Le plan projectif des octaves et les sphères comme espaces homogènes," *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, vol. 230 (1930), pp. 1378-1380.

[2] ———, "Les bouts des espaces homogènes de groupes de Lie," *Annals of Mathematics*, vol. 58 (1953), pp. 443-457.

[3] ——— et J. De Siebenthal, "Les sous-groupes fermés de rang maximum des groupes de Lie clos," *Commentarii Mathematici Helvetici*, vol. 23 (1949), pp. 200-221.

[4] H. Cartan, "La transgression dans un groupe de Lie et dans un espace fibré principal," *Colloque de topologie*, Bruxelles, 1950, Liège et Paris, 1951, 57-71.

[5] S. S. Chern and E. Spanier, "The homology structure of sphere bundles," *Proceedings of the National Academy of Sciences of the U. S. A.*, vol. 36 (1950), pp. 248-255.

[6] B. Eckmann, "Covering spaces and Betti numbers," *Bulletin of the American Mathematical Society*, vol. 55 (1949), pp. 95-101.

[7] J. Leray, "L'homologie d'un espace fibré dont la fibre est connexe," *Journal de Mathématiques pures et appliquées*, vol. 29 (1950), pp. 169-213.

[8] ———, "Sur l'homologie des groupes de Lie, des espaces homogènes et des espaces fibrés principaux," *Colloque de topologie algebrique*, Bruxelles, 1950, Liège et Paris, 1951, 101-115.

[9] D. Montgomery and H. Samuelson, "Transformation groups of spheres," *Annals of Mathematics*, vol. 44 (1943), pp. 454-470.

[10] ——— and L. Zippin, "A clas of transformation groups in E_n," *American Journal of Mathematics*, vol. 63 (1941), pp. 1-8.

[11] ———, *Topological transformation groups*, New York, 1955.

[12] P. Olum, "Non-abelian cohomology and van Kampen's theorem," *Annals of Mathematics*, vol. 68 (1958), pp. 658-668.

[13] H. Samelson, "Beiträge zur Topologie der Gruppenmanniigfaltigkeiten," *ibid.*, vol. 42 (1941), pp. 1091-1137.

[14] H. C. Wang, "Homogeneous spaces with non-vanishing Euler characteristic," *ibid.*, vol. 50 (1949), pp. 925-953.

[15] ———, "Closed manifolds with homogeneous complex structure," *American Journal of Mathematics*, vol. 76 (1954), pp. 1-32.

[16] ———, "A new characterization of spheres of even dimension," *Koninklijke Nederlandse Akademie van Wetenschappen. Proceedings*, vol. 52 (1949), pp. 838-845.

[17] J. Poncet, "Groupes de Lie compacts de transformations de l'espaces euclidien et les sphères comme espaces homogènes," *Commentarii Mathematici Helvetici*, vol. 33 (1959), pp. 109-120.

[18] P. S. Mortert, "On a compact Lie group acting on a sphere" (abstract), *American Mathematical Society Notices*, vol. 5 (1958), p. 498.

[19] N. Nagano, "On some compact transformation groups on spheres," Scientific papers of College of General Education, University of Tokyo, vol. 9 (1959), pp. 213-218.

CONFORMAL TRANSFORMATIONS OF RIEMANN SURFACES.*

By LEON GREENBERG.

1. Introduction. Let $\mathfrak{S}$ be a compact Riemann surface of genus $p > 1$, and let $\boldsymbol{C}(\mathfrak{S})$ be the group of 1-1 conformal transformations of $\mathfrak{S}$ onto itself. It is known that $\boldsymbol{C}(\mathfrak{S})$ is finite and that any finite group is isomorphic to a subgroup of some $\boldsymbol{C}(\mathfrak{S})$ (see [3]). The latter fact can be seen as follows. Let $\boldsymbol{G}$ be a finite group generated by $g_1, g_2, \cdots, g_n$, and let $\boldsymbol{F}$ be the group $\langle a_1, b_1, \cdots, a_n, b_n; \prod_{k=1}^{n} a_k b_k a_k^{-1} b_k^{-1} = 1 \rangle$. There is a homomorphism $\phi: \boldsymbol{F} \to \boldsymbol{G}$, defined by $\phi(a_k) = g_k$, $\phi(b_k) = g_k^{-1}$. If $\boldsymbol{K}$ is the kernel, then $\boldsymbol{G}$ is isomorphic to $\boldsymbol{F}/\boldsymbol{K}$. $\boldsymbol{F}$ is the fundamental group of a compact Riemann surface $\mathfrak{T}$ of genus n. $\mathfrak{T}$ is covered by the half-plane $\mathfrak{H} = \{(x, y) \mid y > 0\}$, and the group of covering transformations is a discrete group of linear fractional transformations, isomorphic to $\boldsymbol{F}$. If we identify points of $\mathfrak{H}$ which are congruent under $\boldsymbol{K}$, we obtain a Riemann surface $\mathfrak{S} = \mathfrak{H}/\boldsymbol{K}$. $\mathfrak{S}$ is compact, since it is a finite covering space of $\mathfrak{T} = \mathfrak{H}/\boldsymbol{F}$ (the index $[\boldsymbol{F}: \boldsymbol{K}] = \text{order}(\boldsymbol{G})$ is finite). A conformal transformation c of $\mathfrak{S}$ can be lifted to a conformal transformation $\bar{c}$ of $\mathfrak{H}$. The transformation $\bar{c}$ maps an orbit $\boldsymbol{K}z$, $z \in \mathfrak{H}$, onto another orbit. From this it follows that $\bar{c}\boldsymbol{K} = \boldsymbol{K}\bar{c}$, or $\bar{c}$ is in the normalizer $\boldsymbol{N}[\boldsymbol{K}, \boldsymbol{C}(\mathfrak{H})]$ of $\boldsymbol{K}$ in $\boldsymbol{C}(\mathfrak{H})$. $\boldsymbol{C}(\mathfrak{S})$ is isomorphic to $\boldsymbol{N}[\boldsymbol{K}, \boldsymbol{C}(\mathfrak{H})]/\boldsymbol{K}$, which contains $\boldsymbol{F}/\boldsymbol{K}$ as a subgroup. Thus $\boldsymbol{G}$ is isomorphic to a subgroup of $\boldsymbol{C}(\mathfrak{S})$.

Since the fundamental groups of the non-compact surfaces are free groups, a similar argument shows that any finite group is realized as a subgroup of $\boldsymbol{C}(\mathfrak{S})$, where $\mathfrak{S}$ is some finite, non-compact Riemann surface (we shall call $\mathfrak{S}$ finite, if it has a finitely generated fundamental group). The purpose of this article is to show:

(a) If $\mathfrak{S}$ is a finite, non-compact Riemann surface which is not homeomorphic to a plane or cylinder, then $\boldsymbol{C}(\mathfrak{S})$ is finite.

(b) If $\boldsymbol{G}$ is a finite group, then there is a finite, non-compact Riemann surface $\mathfrak{S}$, such that $\boldsymbol{C}(\mathfrak{S})$ is isomorphic to $\boldsymbol{G}$. If $\boldsymbol{G}$ is a countable group, then there is a non-compact Riemann surface $\mathfrak{S}$, such that $\boldsymbol{C}(\mathfrak{S})$ is isomorphic to $\boldsymbol{G}$.

* Received September 4, 1959.

2. The hyperbolic plane. The half-plane $\mathfrak{H}$ can be given the Riemannian metric $ds^2 = (dx^2 + dy^2)/y^2$, so that it becomes the Poincaré model of the hyperbolic plane. Let X denote the x-axis, together with the point ∞. The geodesics in $\mathfrak{H}$ are semi-circles and half-lines orthogonal to X. The orientation-preserving isometries are exactly the conformal transformations of $\mathfrak{H}$. These are the linear fractional transformations

$$f(z) = (az + b)/(cz + d),$$

where $ad - bc = 1$ and a, b, c, d are real. We shall often write f as the matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If f is hyperbolic, the geodesic A_f through the fixed points of f is invariant under f, and is called the axis of f. The translation length along A_f is

$$\lambda(f) = 2 \log \tfrac{1}{2}[t + (t^2 - 4)^{\frac{1}{2}}], \tag{1}$$

where $t = a + d$. In later calculations we shall be dealing with the case that $A_f = \{(x, y) \mid x^2 + y^2 = r^2, y > 0\}$. If $\lambda = \lambda(f)$, then

$$f = \begin{pmatrix} \cosh(\lambda/2) & r \sinh(\lambda/2) \\ (1/r)\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}. \tag{2}$$

Corresponding to each isometry f, which moves the point ∞, there are a pair of (circular) geodesics I_f, I_f' which are the isometric circles of f. The transformation f is the product of a reflection in I_f and a reflection in the perpendicular bisector of the line segment between the centers of I_f and I_f'. The isometric circles of f do not intersect, if and only if f is hyperbolic. In this case I_f and I_f' are orthogonal to A_f. When $A_f = \{(x, y) \mid x^3 + y^2 = r^2, y > 0\}$, the isometric circles are of the forms:

$$I_f = \{(x, y) \mid (x + c)^2 + y^2 = s^2, y > 0\},$$
$$I_f' = \{(x, y) \mid (x - c)^2 + y^2 = s^2, y > 0\},$$

where $r^2 + s^2 = c^2$, since I_f and I_f' are orthogonal to A_f. Each of the pairs $\{A_f, \lambda(f)\}$, $\{A_f, I_f\}$ determines f completely. The correspondence between $\lambda = \lambda(f)$ and I_f is given by:

$$\lambda = \log[(c + r)/(c - r)],$$
$$c = r \coth(\lambda/2). \tag{3}$$

(The radius s is determined by: $r^2 + s^2 = c^2$.)

Let I be a geodesic in $\mathfrak{H}$. For each point $z \in \mathfrak{H}$, there is a unique geodesic

I', which passes through z and is orthogonal to I. Let $\text{proj}(z, I) = I \cap I'$. For any set $S \subset \mathfrak{H}$ let $\text{proj}(S, I) = \{\text{proj}(z, I) \mid z \in S\}$. If this latter set has a hyperbolic length, we shall denote this length by $\pi(S, I)$. Now let I and J be geodesics which intersect X in the points $\{i_1, i_2\}$ and $\{j_1, j_2\}$ respectively. It is not hard to show that

$$\pi(I, J) = \pi(J, I) = |\log|R(i_1, i_2; j_1, j_2)||, \tag{4}$$

where $R(i_1, i_2; j_1, j_2)$ is the cross ratio $[(i_1 - j_2)/(i_1 - j_1)][(i_2 - j_1)/(i_2 - j_2)]$. If I is contained in the interior of J or J in the interior of I (that is, if either $j_1 < i_1 < i_2 < j_2$ or $i_1 < j_1 < j_2 < i_2$), then (4) becomes

$$\pi(I, J) = \pi(J, I) = \log R(i_1, i_2; j_1, j_2). \tag{5}$$

3. Discrete groups. Let $\boldsymbol{F}$ be a subgroup of $\boldsymbol{C}(\mathfrak{H})$. The set of limit points of $\boldsymbol{F}$, denoted $L_{\boldsymbol{F}}$, is the intersection with X of the set of limit points of $\{f(z) \mid f \in \boldsymbol{F}\}$, where z is any point in $\mathfrak{H}$. $L_{\boldsymbol{F}}$ is independent of $z \in \mathfrak{H}$ and is invariant under $\boldsymbol{F}$. $L_{\boldsymbol{F}}$ can be the empty set, a single point, two points, X, or a perfect, nowhere dense subset of X. The latter occurs in the case that $\boldsymbol{F}$ is the group of covering transformations corresponding to a non-compact Riemann surface which is not homeomorphic to a plane or cylinder. In this case, $X - L_{\boldsymbol{F}}$ is composed of a countable number of open intervals, called regular intervals. Each regular interval is spanned by a geodesic, which may or may not be an axis of an element in $\boldsymbol{F}$. J. Nielsen [5], [6] has shown that if $\boldsymbol{F}$ is a finitely generated, discrete group, then each of these geodesics is an axis (a bounding axis) and there are a finite number of bounding axes $B_1, B_2, \cdots, B_m$, such that all bounding axes are contained in the set $\{fB_k \mid f \in \boldsymbol{F}, k = 1, 2, \cdots, m\}$.

A discrete group $\boldsymbol{F}$ has a canonical fundamental region, denoted $\mathfrak{R}_{\boldsymbol{F}}$, which consists of the region in $\mathfrak{H}$ outside of all isometric circles of elements in $\boldsymbol{F}$. The following is proved in [4, p. 76].

THEOREM 1. *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be discrete subgroups of $\boldsymbol{C}(\mathfrak{H})$ such that the isometric circles of $\boldsymbol{F}$ are contained in $\mathfrak{R}_{\boldsymbol{G}}$ and the isometric circles of $\boldsymbol{G}$ are contained in $\mathfrak{R}_{\boldsymbol{F}}$. Then the group $\boldsymbol{E}$ generated by $\boldsymbol{F}$ and $\boldsymbol{G}$ is discrete. $\boldsymbol{E}$ is the free product $\boldsymbol{F} * \boldsymbol{G}$ and $\mathfrak{R}_{\boldsymbol{E}} = \mathfrak{R}_{\boldsymbol{F}} \cap \mathfrak{R}_{\boldsymbol{G}}$.*

A method of constructing a class of discrete groups is the following. Let $I_1, I_1', I_2, I_2', \cdots, I_n, I_n'$ be (circular) geodesics which are external to each other and such that $\text{radius}(I_k) = \text{radius}(I_k')$. Let f_k be the linear fractional transformation with isometric circles I_k, I_k'. Let D_k (D_k') be the closed half-disk bounded by X and I_k (I_k'). Theorem 1 shows that the group $\boldsymbol{F}$ generated

by $\{f_k \mid k = 1, 2, \cdots, n\}$ is a discrete, free group of rank n, for which $\mathfrak{R}_F = \mathfrak{H} - \bigcup_{k=1}^{n} (D_k \cup D_k')$. It is not hard to show that the elements of F are all hyperbolic.

We shall need the following definitions. Let F be any group. An N-chain is a sequence of subgroups $F_1, F_2, \cdots, F_n$ such that

(a) $F_k \neq \{1\}$, $k = 1, 2, \cdots, n$,

(b) Either F_k is a normal subgroup of F_{k+1} or F_{k+1} is a normal subgroup of F_k, $k = 1, 2, \cdots, n-1$.

Two subgroups G and H are N-equivalent if there is an N-chain $G = F_1, F_2, \cdots, F_n = H$. G is an N-subgroup if it is N-equivalent to F. G is N-maximal if it contains every subgroup of F which is N-equivalent to G. The following has been proved in [2, Theorem 4].

THEOREM 2. *Let F be a discrete subgroup of $C(\mathfrak{H})$ and let G be a finitely generated N-subgroup of F. Then the index $[F: G]$ is finite.*

LEMMA 1. *Let F be a discrete, non-commutative subgroup of $C(\mathfrak{H})$. Then the normalizer $N[F, C(\mathfrak{H})]$ is discrete.*

Proof. Suppose that there is a sequence $\{n_k\} \subset N[F, C(\mathfrak{H})]$, such that $\lim_{k\to\infty} n_k = 1$. Then $\lim_{k\to\infty} f n_k f^{-1} n_k^{-1} = 1$, where f is any element of F. Since $f n_k f^{-1} n_k^{-1} \in F$, and F is discrete, it follows that $f n_k f^{-1} n_k^{-1} = 1$, for almost all k. Two linear fractional transformations commute, if and only if they have the same fixed points. This shows that F must be a commutative group, contrary to assumption.

THEOREM 3. *Let $\mathfrak{S}$ be a finite Riemann surface which is not homeomorphic to a sphere, plane, cylinder or torus. Then $C(\mathfrak{S})$ is finite.*

Proof. The restrictions on $\mathfrak{S}$ imply that $\mathfrak{S}$ is (conformally) covered by $\mathfrak{H}$. Let G be the group covering transformations and let F be the normalizer $N[G, C(\mathfrak{H})]$. $C(\mathfrak{S})$ is isomorphic to F/G. Since $\mathfrak{S}$ is not homeomorphic to a plane or cylinder, G is non-trivial and non-commutative. Lemma 1 implies that F is discrete, while Theorem 2 implies that the index $[F: G]$ is finite. Therefore $C(\mathfrak{S})$ is finite.

We now wish to show that for any finite group G, there is a finite Riemann surface $\mathfrak{S}$ so that $C(\mathfrak{S})$ is isomorphic to G. In order to prove this, we shall show that for any positive integer $n > 1$, there is an N-maximal, discrete (hyperbolic) group F_n which is a free group of rank n. Suppose, for the

moment, that we know such groups exist. If $\boldsymbol{G}$ is generated by n generators, then $\boldsymbol{G}$ is isomorphic to a factor group $\boldsymbol{F}_n/\boldsymbol{K}$. $\boldsymbol{K}$ is finitely generated, since the index $[\boldsymbol{F}_n : \boldsymbol{K}]$ is finite; therefore the Riemann surface $\mathfrak{S} = \mathfrak{H}/\boldsymbol{K}$ is finite. $\boldsymbol{C}(\mathfrak{S})$ is isomorphic to $\boldsymbol{N}[\boldsymbol{K}, \boldsymbol{C}(\mathfrak{H})]/\boldsymbol{K}$. Now $\boldsymbol{F}_n$ is N-equivalent to $\boldsymbol{N}[\boldsymbol{K}, \boldsymbol{C}(\mathfrak{H})]$. Since $\boldsymbol{F}_n$ is N-maximal, it follows that $\boldsymbol{F}_n = \boldsymbol{N}[\boldsymbol{K}, \boldsymbol{C}(\mathfrak{H})]$, and $\boldsymbol{C}(\mathfrak{S}) \approx \boldsymbol{F}_n/\boldsymbol{K} \approx \boldsymbol{G}$.

Similarly, if we prove the existence of an N-maximal, discrete group, which is free of countably infinite rank, then we can show that any countable group is realized as the conformal group of some Riemann surface.

4. Construction of the group $\boldsymbol{F}_n$. We shall now construct an N-maximal, discrete group $\boldsymbol{F}_n$ which is a free group of rank n. $\boldsymbol{F}_n$ will be generated by elements $f_1, \cdots, f_n$ with isometric circles $I_1, I_1', \cdots, I_n, I_n'$ and axes $A_1, \cdots, A_n$. The bounding axes of $\boldsymbol{F}_n$ will fall into a finite number of disjoint classes $\boldsymbol{F}_nB_1, \boldsymbol{F}_nB_2, \cdots, \boldsymbol{F}_nB_n$. The minimal translation length λ along an axis A is the same for all axes in the class $\boldsymbol{F}_nA$ (since $gA_f = A_{gfg^{-1}}$ and $\lambda(f) = \lambda(gfg^{-1})$). Let λ_k be the minimal translation length for the class $\boldsymbol{F}_nB_k$.

We shall first show that $\boldsymbol{F}_n$ can be constructed so that:

(α) The isometric circles $\{I_k, I_k' \mid k = 1, 2, \cdots, n\}$ are external to each other.

(β) λ_i/λ_j is irrational, if $i \neq j$.

(γ) $\lambda_1 < 2\pi(B_1, B_m)$.

In Section 6 we shall show that the resulting group $\boldsymbol{F}_n$ is N-maximal.

Let us first choose the axes $\{A_k \mid k = 1, 2, \cdots, n\}$, and then choose I_k, I_k' as orthocircles with equal radii, and orthogonal to A_k. We shall choose $A_k = \{(x, y) \mid x^2 + y^2 = r_k^2, y > 0\}$, where $r_1 = 2$, $r_n = 1$ and $\{r_k \mid k = 2, 3, \cdots, n-1\}$ are to be determined, and will satisfy

$$2 = r_1 > r_2 > \cdots > r_n = 1. \tag{6}$$

The isometric circles will be:

$$I_k = \{(x, y) \mid (x + c_k)^2 + y^2 = s_k^2, y > 0\}$$

and

$$I_k' = \{(x, y) \mid (x - c_k)^2 + y^2 = s_k^2, y > 0\},$$

where the $\{c_k\}$ and $\{s_k\}$ are to be determined and will satisfy

$$r_k^2 + s_k^2 = c_k^2, \tag{7}$$

in order that I_k and I_k' be orthogonal to A_k, and

$$(8) \qquad c_1 - s_1 > c_2 + s_2 > c_2 - s_2 > c_3 + s_3 > \cdots > c_n - s_n > 0,$$

in order that the isometric circles be external to each other. For any choice of $\{r_k, c_k, s_k\}$ satisfying (6), (7), (8), the surface $\mathfrak{H}/\boldsymbol{F}_n$ is homeomorphic to a sphere from which $n+1$ closed disks have beeen removed. This means that there are exactly $n+1$ congruence classes of bounding axes, or $m = n+1$. The fundamental region $\mathfrak{R}_{\boldsymbol{F}_n}$ intersects X in the intervals:

$$W_1^{(0)} = X - (-c_1 - s_1, c_1 + s_1),$$
$$V_{-k}^{(0)} = [-c_{k-1} + s_{k-1}, -c_k - s_k], \quad k = 2, 3, \cdots, n,$$
$$V_k^{(0)} = [c_k + s_k, c_{k-1} - s_{k-1}], \quad k = 2, 3, \cdots, n,$$
$$W_{n+1}^{(0)} = [-c_n + s_n, c_n - s_n].$$

Each of these intervals is contained in a regular interval, and any regular interval is congruent to a regular interval which contains at least one of these intervals.

Let $W_1^{(h)} = f_1^h W_1^{(0)}$ $(h = 0, \pm 1, \pm 2, \cdots)$. The interval $W_1 = \bigcup_{h=-\infty}^{\infty} W_1^{(h)}$ is bounded by the limit points $\{-r_1, r_1\}$. Thus $W_1^{(0)}$ is contained in the regular interval W_1, which is spanned by the bounding axis A_1, and we can take $B_1 = A_1$. Similarly, if $W_{n+1}^{(h)} = f_n^h W_{n+1}^{(0)}$, then $W_{n+1} = \bigcup_{h=-\infty}^{\infty} W_{n+1}^{(h)}$ is a regular interval containing $W_{n+1}^{(0)}$, bounded by the limit points $\{-r_n, r_n\}$ and spanned by the bounding axis A_n. We can take $B_{n+1} = A_n$.

Now $f_{k-1}^{-1} V_k^{(0)}$, $V_{-k}^{(0)}$ and $f_k^{-1} V_k^{(0)}$ are three consecutive intervals with common endpoints, and $f_k^{-1} f_{k-1}$ maps the first interval into the third. Therefore if $W_k^{(0)} = V_{-k}^{(0)} \cup f_k^{-1} V_k^{(0)}$ and $W_k^{(h)} = (f_k^{-1} f_{k-1})^h W_k^{(0)}$, then $W_k = \bigcup_{h=-\infty}^{\infty} W_k^{(h)}$ is a regular interval which is spanned by the bounding axis $B_k = A_{f_k^{-1} f_{k-1}}$. Thus we have found representatives for the classes of bounding axes:

$$B_1 = A_1, B_2 = A_{f_2^{-1} f_1}, \cdots, B_n = A_{f_n^{-1} f_{n-1}}, B_{n+1} = A_n.$$

We are now ready to begin choosing the f_k so that conditions (α), (β) and (γ) are satisfied. We first choose f_1 so that $\lambda(f_1) < 2\pi(A_1, A_n)$ and the points -1 and 1 are outside the isometric circles I_1, I_1'. This latter condition is exactly $\lambda(f_1) > \pi(A_1, A_n)$. Since $\pi(A_1, A_n) = \log R(-1, 1; -2, 2) = \log 9$, the condition on f_1 is

$$\log 9 < \lambda(f_1) < 2 \log 9.$$

Choose any number λ between $\log 9$ and $2\log 9$ and define

$$f_1 = \begin{pmatrix} \cosh(\lambda/2) & 2\sinh(\lambda/2) \\ \frac{1}{2}\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}.$$

Then $\lambda = \lambda(f_1) = \lambda_1$ satisfies condition (γ). Furthermore the condition $\lambda(f_1) > \pi(A_1, A_n)$ guarantees that $c_1 - s_1 > 1$.

Now suppose that we have found $f_1, f_2, \cdots, f_k$ $(k \leqq n-2)$ such that

$$c_1 - s_1 > c_2 + s_2 > c_2 - s_2 > \cdots > c_k - s_k > 1, \qquad 2 > r_2 > r_3 > \cdots > 1,$$

and such that the ratio of any two of the numbers

$$\lambda(f_1), \lambda(f_2^{-1}f_1), \cdots, \lambda(f_k^{-1}f_{k-1})$$

is irrational. We wish to extend these conditions to a transformation f_{k+1}. The condition $c_k - s_k > c_{k+1} + s_{k+1}$ is equivalent to

$$c_{k+1} < [(c_k - s_k)^2 + r_{k+1}^2]/2(c_k - s_k).$$

The condition $c_{k+1} - s_{k+1} > 1$ is equivalent to

$$c_{k+1} < (1 + r_{k+1}^2)/2.$$

Choose r_{k+1} so that $1 < r_{k+1} < (c_k - s_k)^{\frac{1}{2}}$. Then it follows that

$$(1 + r_{k+1}^2)/2 < [(c_k - s_k)^2 + r_{k+1}^2]/2(c_k - s_k) \text{ and } r_{k+1} < r_k.$$

Now if we choose any c_{k+1} so that

$$r_{k+1} < c_{k+1} < (1 + r_{k+1}^2)/2, \tag{9}$$

and define $s_{k+1} = (c_{k+1}^2 - r_{k+1}^2)^{\frac{1}{2}}$, it will follow that

$$c_k - s_k > c_{k+1} + s_{k+1} > c_{k+1} - s_{k+1} > 1.$$

However we wish to choose c_{k+1} so that the ratio of any two of the numbers $\lambda(f_1), \lambda(f_2^{-1}f_1), \cdots, \lambda(f_{k+1}^{-1}f_k)$ is irrational. The relations (3) show that (9) is equivalent to

$$2\log[(r_{k+1}+1)/(r_{k+1}-1)] < \lambda(f_{k+1}) < \infty. \tag{10}$$

Let $f(x) = \begin{pmatrix} \cosh(x/2) & r_{k+1}\sinh(x/2) \\ (1/r_{k+1})\sinh(x/2) & \cosh(x/2) \end{pmatrix}$ and $\psi(x) = \lambda(f(x)^{-1}f_k)$.

The function ψ is continuous and maps the interval

$$I = (2\log[(r_{k+1}+1)/(r_{k-1}-1)], \infty)$$

onto an interval J. Let

$$K_1 = \{r\lambda(f_1) \mid r \text{ rational}\}, \qquad K_i = \{r\lambda(f_i^{-1}f_{i-1}) \mid r \text{ rational}\},$$
$$i = 2, 3, \cdots, k.$$

The set $K = \bigcup_{i=1}^{k} K_i$ is countable, so $J - K$ is non-empty. Choose $x_0 \in I$ so that $\psi(x_0) \in J - K$, and define $f_{k+1} = f(x_0)$. The transformations $f_1, f_2, \cdots, f_{k+1}$ satisfy the conditions:

$$c_1 - s_1 > c_2 + s_2 > c_2 - s_2 > \cdots > c_{k+1} - s_{k+1} > 1,$$
$$2 = r_1 > r_2 > \cdots > r_{k+1} > 1,$$

and the ratio of any two of the numbers $\lambda(f_1), \lambda(f_2^{-1}f_1), \cdots, \lambda(f_{k+1}^{-1}f_k)$ is irrational.

Now suppose we have found $f_1, f_2, \cdots, f_{n-1}$ with the required properties. We wish to find f_n so that $r_n = 1$, $c_{n-1} - s_{n-1} > c_n + s_n > c_n - s_n > 0$ and condition (β) are fulfilled. The second condition is equivalent to

$$1 < c_n < u = [(c_{n-1} - s_{n-1})^2 + 1]/2(c_{n-1} - s_{n-1}),$$

which, as before, is equivalent to

$$\log[(u+1)/(u-1)] < \lambda(f_n) < \infty.$$

Let
$$f(x) = \begin{pmatrix} \cosh(x/2) & \sinh(x/2) \\ \sinh(x/2) & \cosh(x/2) \end{pmatrix}$$

and
$$\psi(x) = \lambda(f(x)^{-1}f_{n-1})/x.$$

The function ψ is continuous on the interval $I = (\log[(u+1)/(u-1)], \infty)$ and maps I onto an interval J. Define K_i as before and $K = \bigcup_{i=1}^{n-1} K_i$. Now the set $M = \{x \mid x \in I \text{ and } \psi(x) \text{ is irrational}\}$ is an uncountable set, while K is countable. Therefore M cannot be a subset of K, and $M \cap (I - K)$ is non-empty. Choose $x_0 \in M \cap (I - K)$ and let $f_n = f(x_0)$. Then the transformations $\{f_k \mid k = 1, 2, \cdots, n\}$ satisfy conditions (α), (β), (γ).

5. Construction of the group F_∞. In this section we shall construct a discrete group F_∞ which is a free group of rank $\aleph_0$, and which will later be shown to be N-maximal. The construction will be similar to that in the previous section. F_∞ will be generated by transformations $\{f_k \mid k = 0, 1, 2, \cdots\}$, with isometric circles $\{I_k, I_k' \mid k = 0, 1, 2, \cdots\}$ and axes $\{A_k \mid k = 0, 1, 2, \cdots\}$. We can choose $A_k = \{(x, y) \mid x^2 + y^2 = r_k^2, y > 0\}$, where $r_0 = \frac{1}{2}$, $r_1 = 2$, and $r_1 > r_2 > r_3 > \cdots > 1$. The isometric circles will be

$$I_k = \{(x, y) \mid (x + c_k)^2 + y^2 = s_k^2, y > 0\},$$
$$I_k' = \{(x, y) \mid (x - c_k)^2 + y^2 = s_k^2, y > 0\},$$

where $r_k^2 + s_k^2 = c_k^2$,

$$c_1 - s_1 > c_2 + s_2 > c_2 - s_2 > \cdots > c_k + s_k > c_k - s_k > \cdots$$
$$> 1 > c_0 + s_0 > c_0 - s_0 > 0,$$

and $\lim_{k\to\infty} c_k = 1$.

For any choice of $\{r_k, c_k, s_k\}$, satisfying the above conditions, the fundamental region $\mathfrak{R}_{F_\infty}$ intersects X in the intervals:

$$W_0^{(0)} = [-c_0 + s_0, c_0 - s_0],$$
$$W_1^{(0)} = X - (-c_1 - s_1, c_1 + s_1),$$
$$W_- = [-1, -c_0 - s_0],$$
$$W_+ = [c_0 + s_0, 1],$$
$$V_{-k}^{(0)} = [-c_{k-1} + s_{k-1}, -c_k - s_k], \quad (k = 2, 3, \cdots),$$
$$V_k^{(0)} = [c_k + s_k, c_{k-1} - s_{k-1}], \quad (k = 2, 3, \cdots).$$

Any regular interval is congruent, under F_∞, to an interval which contains at least one of the above intervals.

Let $W_0^{(h)} = f_0^h W_0^{(0)}$, $W_0 = \bigcup_{h=-\infty}^{\infty} W_0^{(h)}$, $W_1^{(h)} = f_1^h W_1^{(0)}$ and $W_1 = \bigcup_{h=-\infty}^{\infty} W_1^{(h)}$. Then W_0 and W_1 are regular intervals, spanned by the bounding axes $B_0 = A_0$, $B_1 = A_1$, respectively. Let $W_k^{(0)} = V_{-k}^{(0)} \cup f_k^{-1} V_k^{(0)}$, $W_k^{(h)} = (f_k^{-1} f_{k-1})^h W_k^{(0)}$ and $W_k = \bigcup_{h=-\infty}^{\infty} W_k^{(h)}$. Then W_k is a regular interval, spanned by the bounding axis $B_k = A_{f_k^{-1} f_{k-1}}$. The interval $W = W_- \cup f_0^{-1} W_+$ is a regular interval, bounded by the limit points $\{-1, f_0^{-1}(1)\}$. The geodesic which spans W is not an axis of F_∞. Every regular interval is contained in the set $\{fW, fW_k | f \in F_\infty, k = 0, 1, 2, \cdots\}$, and every bounding axis is contained in $\{fB_k | f \in F_\infty, k = 0, 1, 2, \cdots\}$. Let $\lambda_0 = \lambda(f_0)$, $\lambda_1 = \lambda(f_1)$ and $\lambda_k = \lambda(f_k^{-1} f_{k-1})$ $(k = 2, 3, \cdots)$. In the same way as in the case of $\boldsymbol{F}_n$, we can show that $\{f_k | k = 0, 1, 2, \cdots\}$ can be chosen so that:

(α) The isometric circles $\{I_k, I_k' | k = 0, 1, 2, \cdots\}$ are external to each other.

(β) λ_i/λ_j is irrational, if $i \neq j$.

(γ) $\lambda_1 < 2\pi(B_1, B_0)$.

6. The N-maximality of $\boldsymbol{F}_n$ and $\boldsymbol{F}_\infty$. The following is proved in [1, p. 43].

LEMMA 2. *Let $\boldsymbol{G}$ be a discrete subgroup of $\boldsymbol{C}(\mathfrak{H})$. If Y is a closed subset of X, which contains more than one point and is invariant under $\boldsymbol{G}$, then $Y \supset L_{\boldsymbol{G}}$.*

LEMMA 3. *Let $\boldsymbol{E}$ and $\boldsymbol{F}$ be subgroups of $\boldsymbol{C}(\mathfrak{H})$. Suppose that $\boldsymbol{F}$ is discrete, $L_{\boldsymbol{F}}$ contains more than two points*[1] *and $\boldsymbol{E}$ is N-equivalent to $\boldsymbol{F}$. Then $L_{\boldsymbol{E}} = L_{\boldsymbol{F}}$.*

Proof. There is an N-chain $\boldsymbol{F} = G_1, \boldsymbol{G}_2, \cdots, \boldsymbol{G}_m = \boldsymbol{E}$. We shall prove inductively that $\boldsymbol{G}_k$ is discrete and $L_{\boldsymbol{G}_k} = L_{\boldsymbol{F}}$.

Let us assume the above is true for $\boldsymbol{G}_k$, and prove it for $\boldsymbol{G}_{k+1}$. Either $\boldsymbol{G}_k$ is a normal subgroup of $\boldsymbol{G}_{k+1}$ or $\boldsymbol{G}_{k+1}$ is a normal subgroup of $\boldsymbol{G}_k$. We shall first assume the former. Then Lemma 1 implies that $\boldsymbol{G}_{k+1}$ is discrete. Since $\boldsymbol{G}_{k+1} \supset \boldsymbol{G}_k$, $L_{\boldsymbol{F}} = L_{\boldsymbol{G}_k} \subset L_{\boldsymbol{G}_{k+1}}$. We need to show that $L_{\boldsymbol{G}_{k+1}} \subset L_{\boldsymbol{G}_k}$. To do this, we shall show that $L_{\boldsymbol{G}_k}$ is invariant under $\boldsymbol{G}_{k+1}$.

If $z_0 \in L_{\boldsymbol{G}_k}$, there is a sequence $\{g_m\} \subset \boldsymbol{G}_k$, so that $\lim_{m\to\infty} g_m(z) = z_0$, for any point $z \in \mathfrak{H}$. Let $g \in \boldsymbol{G}_{k+1}$. Then $\lim_{m\to\infty} g_m g^{-1}(z) = z_0$ and $\lim_{m\to\infty} g g_m g^{-1}(z) = g(z_0)$, since g is continuous. But $g g_m g^{-1} \in \boldsymbol{G}_k$, so $g(z_0) \in L_{\boldsymbol{G}_k}$, and $L_{\boldsymbol{G}_k}$ is invariant under $\boldsymbol{G}_{k+1}$. Furthermore $L_{\boldsymbol{G}_k} = L_{\boldsymbol{F}}$ contains more than one point. Lemma 2 (with $Y = L_{\boldsymbol{G}_k}$ and $\boldsymbol{G} = \boldsymbol{G}_{k+1}$) now implies that $L_{\boldsymbol{G}_k} \subset L_{\boldsymbol{G}_{k+1}}$. Therefore $L_{\boldsymbol{G}_{k+1}} = L_{\boldsymbol{G}_k} = L_{\boldsymbol{F}}$.

Now suppose that $\boldsymbol{G}_{k+1}$ is a normal subgroup of $\boldsymbol{G}_k$. It is immediate that $\boldsymbol{G}_{k+1}$ is discrete and $L_{\boldsymbol{G}_{k+1}} \subset L_{\boldsymbol{G}_k}$. The argument given above (reversing the roles of $\boldsymbol{G}_k$ and $\boldsymbol{G}_{k+1}$) shows that $L_{\boldsymbol{G}_{k+1}}$ is invariant under $\boldsymbol{G}_k$. If $L_{\boldsymbol{G}_{k+1}}$ is the empty set or $L_{\boldsymbol{G}_{k+1}}$ consists of one point, then $\boldsymbol{G}_{k+1}$ is an elliptic or parabolic group. In this case there is a point $z \in \mathfrak{H}$ or X, which is the common fixed point (in $\mathfrak{H}$ or X) of all elements in $\boldsymbol{G}_{k+1}$. If $g \in \boldsymbol{G}_k$ and $h \in \boldsymbol{G}_{k+1}$, then $g(z)$ is a fixed point of $ghg^{-1} \in \boldsymbol{G}_{k+1}$. It follows that z must be a common fixed point for all elements of $\boldsymbol{G}_k$. Then $\boldsymbol{G}_k$ is an elliptic or parabolic group. This contradicts the assumption that $L_{\boldsymbol{G}_k}$ $(= L_{\boldsymbol{F}})$ contains more than two points. Therefore $L_{\boldsymbol{G}_{k+1}}$ contains more than one point, and Lemma 2 implies that $L_{\boldsymbol{G}_{k+1}} \supset L_{\boldsymbol{G}_{k+1}}$. It follows that $L_{\boldsymbol{G}_{k+1}} = L_{\boldsymbol{G}_k}$ and $L_{\boldsymbol{E}} = L_{\boldsymbol{F}}$.

Let $\boldsymbol{G}_n = \{g \mid g \in \boldsymbol{C}(\mathfrak{H})$ and $g L_{\boldsymbol{F}_n} = L_{\boldsymbol{F}_n}\}$ and $G_\infty = \{g \mid g \in \boldsymbol{C}(\mathfrak{H})$ and $g L_{\boldsymbol{F}_\infty} = L_{\boldsymbol{F}_\infty}\}$.

LEMMA 4. *$\boldsymbol{G}_n$ and $\boldsymbol{G}_\infty$ are discrete.*

Proof. A closed subgroup of $\boldsymbol{C}(\mathfrak{H})$ must be one of the following: 1) the group of all hyperbolic transformations with a given pair of fixed points in

[1] The Lemma is true without this assumption, but we are interested only in this case.

X; 2) the group of all parabolic transformations with a given fixed point in X; 3) the group of all elliptic transformations with a given fixed point in $\mathfrak{H}$; 4) the group of all hyperbolic and parabolic transformations with a given fixed point in X; 5) the group of all hyperbolic and elliptic transformations which leave a given geodesic in $\mathfrak{H}$ invariant; 6) $\boldsymbol{C}(\mathfrak{H})$; 7) a discrete group.

The only possibility for the closures $\bar{\boldsymbol{G}}_n$ and $\bar{\boldsymbol{G}}_\infty$ is 7). Therefore $\boldsymbol{G}_n$ and $\boldsymbol{G}_\infty$ are discrete.

LEMMA 5. *$\boldsymbol{G}_n = \boldsymbol{F}_n$ and $\boldsymbol{G}_\infty = \boldsymbol{F}_\infty$.*

Proof. It is clear that $\boldsymbol{F}_n \subset \boldsymbol{G}_n$ and $\boldsymbol{F}_\infty \subset \boldsymbol{G}_\infty$, since $L_{\boldsymbol{F}}$ is invariant under $\boldsymbol{F}$, for any group $\boldsymbol{F} \subset \boldsymbol{C}(\mathfrak{H})$. We must prove that $\boldsymbol{F}_n \supset \boldsymbol{G}_n$ and $\boldsymbol{F}_\infty \supset \boldsymbol{G}_\infty$. We shall first consider the case of $\boldsymbol{F}_n$.

Let $g \in \boldsymbol{G}_n$. The regular interval W_1, spanned by B_1, is mapped by g onto another regular interval Y, spanned by an axis $A_{f'} = gA_1$, where $f' \in \boldsymbol{F}_n$. Now Y is in one of the classes $\{\boldsymbol{F}_n W_k \mid k = 1, 2, \cdots, n+1\}$. Suppose Y is in $\boldsymbol{F}_n W_k$, so that $gW_1 = fW_k$ and $gB_1 = fB_k$, where $f \in \boldsymbol{F}_n$. Let g_1, g_k be primary elements of $\boldsymbol{G}_n$ with axes B_1, B_k respectively (such elements exist since $\boldsymbol{G}_n$ is discrete). Then since B_1 and B_k are congruent in $\boldsymbol{G}$, it follows that $\lambda(g_1) = \lambda(g_k)$. Furthermore $\lambda_1 = \lambda(f_1)$ must be an integral multiple of $\lambda(g_1)$, say $\lambda_1 = p\lambda(g_1)$. Similarly $\lambda_k = \lambda(f_k^{-1} f_{k-1}) = q\lambda(g_k)$. Therefore $\lambda_1/\lambda_k = p/q$, which is contrary to the construction of $\boldsymbol{F}_n$, unless $k = 1$. Thus, we have found that $gB_1 = fB_1$ (and $gW_1 = fW_1$). The transformation $f^{-1}g$ maps W_1 onto itself, and is therefore a hyperbolic transformation with axis B_1. Therefore $f^{-1}g = g_1^k$, for some integer k, or $g = fg_1^k$. This implies that the index $[\boldsymbol{G}_n : \boldsymbol{F}_n]$ is finite, so $\boldsymbol{G}_n$ is N-equivalent to $\boldsymbol{F}_n$. Therefore, according to Lemma 3, $L_{\boldsymbol{F}_n} = L_{\boldsymbol{G}_n}$.

We now note that the condition $\lambda_1 < 2\pi(B_1, B_{n+1})$ implies that $\lambda(g_1) = \lambda(f_1)$, or $g_1 = f_1^{\pm 1}$. For suppose g is a transformation with axis B_1 such that $\lambda(g) < \pi(B_1, B_{n+1})$. Then B_{n+1} intersects $g(B_{n+1})$, so that $gf_{n+1}g^{-1}$ has a fixed point (which is a limit point) in the regular interval W_{n+1}. Since W_{n+1} is a regular interval for $\boldsymbol{G}_n$ (for $L_{\boldsymbol{F}_n} = L_{\boldsymbol{G}_n}$), no such element g can belong to $\boldsymbol{G}_n$. Now $\lambda(f_1) = p\lambda(g_1)$, for some integer p. If $p > 1$, then $\lambda(g_1) = \lambda(f_1)/p \leqq \lambda(f_1)/2 < \pi(B_1, B_{n+1})$. We conclude that $g_1 = f_1^{\pm 1}$. Since for any $g \in \boldsymbol{G}_n$ there is $f \in \boldsymbol{F}_n$ and an integer k so that $g = fg_1^k$, it now follows that $\boldsymbol{G}_n = \boldsymbol{F}_n$.

The argument for $\boldsymbol{F}_\infty$ is similar. The only difference is that gW_1 might be fW rather than fW_k. This can happen only if the geodesic, spanning W, is an axis of an element in $\boldsymbol{G}_\infty$ (even though it is not an axis of an element in $\boldsymbol{F}_\infty$). We shall show that this cannot happen, by showing that the index

$r = [\boldsymbol{G}_\infty : \boldsymbol{F}_\infty]$ is finite. If this is true, and $g \in \boldsymbol{G}_\infty$, then one of the elements $g, g^2, \cdots, g^r$ must be in $\boldsymbol{F}_\infty$. These elements all have the same axis, so we see that for any $g \in \boldsymbol{G}_\infty$ there is an element in $\boldsymbol{F}_\infty$ whose axis coincides with A_g.

If W_1 is not congruent to W under $\boldsymbol{G}_\infty$, then the argument used for $\boldsymbol{F}_n$ is valid here. Suppose then, that W_1 is congruent to W. W_2 cannot be congruent to W under $\boldsymbol{G}_\infty$. For in this case W_2 would be congruent to W_1, and λ_2/λ_1 would be rational.

An element $g \in \boldsymbol{G}_\infty$ maps W_2 onto a regular interval W', which is in some class $\boldsymbol{F}_\infty W_k$. We now have $gW_2 = fW_k$, where $f \in \boldsymbol{F}_\infty$. By the same argument as before, we see that $k = 2$, because λ_2/λ_k is irrational. Therefore $gW_2 = fW_2$ and $f^{-1}g$ maps W_2 onto itself. If g_2 is a primary element in $\boldsymbol{G}_\infty$ with axis B_2, then $f^{-1}g = g_2^m$. If $\lambda_2 = n\lambda(g_2)$, then every element in $\boldsymbol{G}_\infty$ is congruent mod $\boldsymbol{F}_\infty$ to one of the elements $1, g_2, \cdots, g_2^{n-1}$. Thus the index $[\boldsymbol{G}_\infty : \boldsymbol{F}_\infty]$ is finite, and as we have seen, this implies that $\boldsymbol{G}_\infty = \boldsymbol{F}_\infty$.

THEOREM 4. *The groups $\boldsymbol{F}_n$ and $\boldsymbol{F}_\infty$ are N-maximal.*

Proof. Let $\boldsymbol{F}$ be one of the groups $\boldsymbol{F}_n$ or $\boldsymbol{F}_\infty$. If $\boldsymbol{E}$ is N-equivalent to $\boldsymbol{F}$, Lemma 3 implies that $L_{\boldsymbol{E}} = L_{\boldsymbol{F}}$. Therefore $L_{\boldsymbol{F}}$ is invariant under $\boldsymbol{E}$, and $\boldsymbol{E} \subset \boldsymbol{G} = \{g \mid g \in \boldsymbol{C}(\mathfrak{S}) \text{ and } gL_{\boldsymbol{F}} = L_{\boldsymbol{F}}\}$. But according to Lemma 5, $\boldsymbol{G} = \boldsymbol{F}$. Therefore $\boldsymbol{E} \subset \boldsymbol{F}$, and $\boldsymbol{F}$ is N-maximal.

The previous theorem and the remarks at the end of Section 3 now imply the following.

THEOREM 5. *Let $\boldsymbol{G}$ be a countable group. Then there is a Riemann surface $\mathfrak{S}$, such that $\boldsymbol{C}(\mathfrak{S})$ is isomorphic to $\boldsymbol{G}$. If $\boldsymbol{G}$ is finite, the surface $\mathfrak{S}$ can be taken to be finite.*

BROWN UNIVERSITY.

REFERENCES.

[1] L. Ford, *Automorphic Functions*, New York, 1951.

[2] L. Greenberg, "Discrete groups of motions," *Canadian Journal of Mathematics*, vol. 12 (1960), pp. 415-426.

[3] A. Hurwitz, "Ueber algebraische Gebilde mit eindeutigen Transformationen in sich," *Mathematische Annalen*, Band 41 (1893), pp. 403-442.

[4] W. Magnus, *Discrete groups*, New York University notes, 1952.

[5] J. Nielsen, "Über Gruppen linearer Transformationen," *Mittelungen der Mathematischen Gesellschaft in Hamburg*, Band VIII (1940), pp. 82-104.

[6] ———, "Nogle grundlaegende begreber vedrørende diskontinuerte grupper af lineaere substitutioner i en kompleks variabel," *Den 11te Skandinaviske Matematikerkongress i Trondheim* (1949), pp. 61-70.

SUBGROUPS OF THE MODULAR GROUP AND SUMS OF SQUARES.*[1]

By MORRIS NEWMAN.

Introduction. The problem of counting the number of representations $r_s(n)$ of a positive integer n as the sum of a fixed number of squares s has been of interest to mathematicians for many years. This function is defined as follows: If n is a non-negative integer, then $r_s(n)$ is the coefficient of q^n in

$$\{\sum_{n=-\infty}^{\infty} q^{n^2}\}^s;$$

otherwise, $r_s(n)$ is zero. The results obtained have generally been of two kinds: Asymptotic formulas, showing the behaviour of $r_s(n)$ for large n and fixed s; and exact formulas, which express $r_s(n)$ in terms of other arithmetical functions such as divisor sums and which are considered to be more elementary than $r_s(n)$. There is yet a third kind of relationship which has occurred occasionally in the literature but has not been fully exploited. This is the recurrence formula, an instance of which is

$$r_3(np^2) - r_3(n) = (p - (-n/p))(r_3(n) - r_3(n/p^2)),$$

where p is an odd prime, n an arbitrary integer and $(-n/p)$ the usual quadratic reciprocity symbol. Such formulas give information about the internal structure of the numbers $r_s(n)$.

The principal purpose of this paper is to prove that recurrence formulas of this type exist for all positive integers s and all odd primes p. The number of terms in these formulas depends only on s and p. The method used is the 'subgroup' method which concerns itself with functions invariant with respect to the substitutions of a suitable subgroup of the modular group Γ, regarded as linear fractional transformations. Related problems have been treated by the author in [6] and [10] by the same method.

The paper falls naturally into three parts. In § 1 the necessary group-

* Received October 9, 1959.

[1] The preparation of this paper was supported (in part) by the Office of Naval Research.

theoretic preliminaries are developed. These are given in much greater detail than is actually required for the remainder of the paper. In addition several theorems relevant to the structure of modular subgroups are proved which are not used in the remainder of the paper. These theorems are of high intrinsic interest however and generalize previous theorems given in [8] and [9]. In § 2 the necessary transformation equations for the associated modular forms are derived. In § 3 the class of modular functions to be studied is defined and the properties of these functions worked out, leading to a proof of the theorem mentioned above. Finally, it is shown how a Hauptmodul for a certain subgroup of Γ can be constructed from these functions.

1. Subgroups of the modular group. In what follows Γ is the full modular group; that is, the totality of rational integral 2×2 matrices of determinant 1. Γ is generated by the matrices

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

with defining relations

$$T^2 = (ST)^3 = -I,$$

where I is the identity matrix. We also define

$$W = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then for all integral k,

$$S^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \qquad W^k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

Furthermore

$$W^{-1}T = TS, \qquad W = STS.$$

The group generated by the elements A, B of Γ will be denoted by $[A, B]$. If A is the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and τ is a complex number, then by $A\tau$ we shall understand

$$(a\tau + b)/(c\tau + d).$$

We shall say that $g(\tau)$ is a function on a subgroup G of Γ, if $g(A\tau) = g(\tau)$ for all $A \in G$.

Let G be a subgroup of Γ. We define a class of congruence subgroups $G(n)$, n a positive integer, as follows: The element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of G belongs to $G(n)$ if and only if $c \equiv 0 \pmod{n}$. If $G = \Gamma$, then the groups so defined become the customary congruence subgroups $\Gamma_0(n)$. We shall assume that G does not consists solely of powers of S, since then $G(n) = G$ and the groups are without interest. If m and n are positive integers and $[m, n]$ is their least common multiple, then

$$G(m) \cap G(n) = G([m, n]). \tag{1.1}$$

Formula (1.1) implies that if $m \mid n$, then $G(m) \supset G(n)$. The converse is not necessarily true. We can say however that if $G(m) \supset G(n)$, then $m \mid n\delta$, where δ is the greatest common divisor of all integers c such that

$$\begin{bmatrix} a & b \\ nc & d \end{bmatrix} \in G(n).$$

For example, suppose that $m > 1$, $(m, n) = 1$ and take $G = \Gamma_0(m)$. Then $G(m) = \Gamma_0(m)$, $G(n) = \Gamma_0(mn)$. Thus $G(m) \supset G(n)$ but m does not divide n. Here $\delta = m$ and certainly $m \mid n\delta$.

We now define the integers α, β as follows: If some non-zero power of S is in G, let α be the least positive integer such that $S^\alpha \in G$; otherwise let $\alpha = 0$. If some nan-zero power of W is in G, let β be the least positive integer such that $W^\beta \in G$; otherwise let $\beta = 0$. We say that G is of type (α, β). For example, $\Gamma_0(n)$ is of type $(1, n)$, $\Gamma^0(n) = T^{-1}\Gamma_0(n)T$ is of type $(n, 1)$, $\Gamma_0^0(n) = \Gamma_0(n) \cap \Gamma^0(n)$ is of type (n, n) and the principal congruence subgroup of level n (which we denote by $\Gamma[n]$) is of type (n, n).

Every group of type (α, β) contains as a subgroup the group

$$G_{\alpha,\beta} = [S^\alpha, W^\beta];$$

and it is of interest to investigate these groups further. We have

THEOREM 1.1. *Suppose that α, β are positive. Then if $\alpha\beta > 4$, $G_{\alpha,\beta}$ is of infinite index in Γ. If $\alpha\beta \leqq 4$, $G_{\alpha,\beta}$ is of finite index in Γ and $G_{1,1} = \Gamma$, $G_{1,2} = \Gamma_0(2)$, $G_{1,3} = \Gamma_0(3)$, $G_{1,4} = \Gamma_0(4)$, $G_{2,1} = \Gamma^0(2)$, $G_{2,2} = \Gamma[2]$, $G_{3,1} = \Gamma^0(3)$, $G_{4,1} = \Gamma^0(4)$. Furthermore $G_{\alpha,\beta}$ is a free group for $\alpha\beta \geqq 4$.*

Proof. If $\alpha\beta > 4$, the fundamental region of $G_{\alpha,\beta}$ contains a segment of the real axis, which proves that $G_{\alpha,\beta}$ is of infinite index in Γ. Also it has

been shown in [1] and [3] that for $\alpha\beta \geqq 4$, $G_{\alpha,\beta}$ is a free group. We now suppose $\alpha\beta \leqq 4$. Because of the relationship

$$T^{-1}G_{\alpha,\beta}T = G_{\beta,\alpha}$$

we need only consider $\alpha \leqq \beta$. The group $G_{1,1}$ is clearly Γ, since $T = S^{-1}WS^{-1}$. Suppose $2 \leqq n \leqq 4$. We have that $\Gamma_0(n) \supset G_{1,n}$. Let

$$A = \begin{bmatrix} a & b \\ nc & d \end{bmatrix} \in \Gamma_0(n).$$

Then

$$W^{nx}A = \begin{bmatrix} a & * \\ nc_0 & * \end{bmatrix}, \qquad c_0 = ax + c.$$

Since a cannot vanish, x can be determined so that $|c_0| \leqq \frac{1}{2}|a|$. Furthermore

$$S^{y}W^{nx}A = \begin{bmatrix} a_0 & * \\ nc_0 & * \end{bmatrix}, \qquad a_0 = a + nc_0y.$$

Thus if $c_0 \neq 0$, y can be determined so that $|a_0| \leqq \frac{1}{2}n|c_0|$. We can conclude therefore that either

$$c_0 = 0 \tag{1.2}$$

or integers x, y can be determined so that $|a_0| \leqq \frac{1}{4}n|a| \leqq |a|$. If $c_0 = 0$, then $W^{nx}A$ is a power of S and so $A \in G_{1,n}$. If $c_0 \neq 0$, then $|a_0| \leqq |a|$. If $|a_0| = |a|$, then the inequalities

$$|c_0| \leqq \tfrac{1}{2}|a| = \tfrac{1}{2}|a_0| \leqq \tfrac{1}{4}n|c_0| \leqq |c_0|$$

imply $a_0 = \pm 2$, $c_0 = \pm 1$, $n = 4$. But this is impossible, since $(a_0, nc_0) = 1$. Thus if $c_0 \neq 0$, then $|a_0| < |a|$. Therefore the process can be repeated until (1.2) holds. Thus $A \in G_{1,n}$, $\Gamma_0(n) \subset G_{1,n}$ and so $\Gamma_0(n) = G_{1,n}$.

Finally, put

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then it is easy to verfy that

$$G_{2,2} = D^{-1}G_{1,4}D = D^{-1}\Gamma_0(4)D = \Gamma[2].$$

This completes the proof of the theorem.

Remark. We have implicitly identified a matrix with its negative in this theorem, but we keep the distinction elsewhere.

An interesting application of Theorem 1.1 is

THEOREM 1.2. *Let α, β be positive and suppose G is a group of type (α, β) with $\alpha\beta \leqq 4$. Then G is of finite index in Γ.*

It is not immediately obvious that there are non-abelian groups of type $(0,0)$. An example is furnished by the following theorem:

THEOREM 1.3. *There are non-abelian subgroups of Γ of type $(0,0)$.*

Proof. Put

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and consider $[A, T]$. Since A is not of period 2 and $AT = TA^{-1}$, $[A, T]$ is non-abelian. Since $T^2 = -I$, every element of $[AT]$ is of one of the forms $\pm A^k$, $\pm TA^k$, k an integer. But A^k contains a zero entry if and only if $k = 0$, so that $\pm I$, $\pm T$ are the only elements of $[A, T]$ containing zero entries. This proves the theorem.

The previous construction can be generalized. It is not difficult to show that if A and B are elements of Γ such that $AB = BA^{-1}$ and A is not scalar, then both B and AB are similar to T over Γ. Further, essentially the same conclusion can be reached for $AB = BC$, if it is assumed that C is a polynomial in A.

We assume from now on that G is of type (α, β) with $\alpha\beta \neq 0$. We are interested in proving inclusion theorems for the groups $G(n)$ and for certain other groups, and we require the following result:

LEMMA 1.1. *Suppose that m, n are positive integers and that $(m, \alpha\beta) = 1$. Then S^α, $W^{n\beta}$ generate the coset representatives (left or right) of $G(n)$* modulo $G(mn)$.

Proof. Let

$$M = \begin{bmatrix} a & b \\ nc & d \end{bmatrix} \in G(n).$$

It suffices to show that integers x, y can be found such that $W^{xn\beta}S^{y\alpha}M \in G(mn)$. Consider

$$S^{y\alpha}M = \begin{bmatrix} a + \alpha ncy & * \\ nc & * \end{bmatrix}.$$

We have $\Delta = (a, \alpha nc) = (a, \alpha)$ since M is unimodular. Thus $\Delta \mid \alpha$ and since $(\alpha, m) = 1$, $(\Delta, m) = 1$ also. Thus y may be found so that $(a + \alpha ncy, m) = 1$. For such a y, put $a_0 = a + \alpha ncy$. Then

$$W^{xn\beta}S^{y\alpha}M = \begin{bmatrix} * & * \\ n(\beta a_0 x + c) & * \end{bmatrix}.$$

Since $(\beta a_0, m) = 1$, x can be determined so that $\beta a_0 x + c \equiv 0 \pmod{m}$. For this choice of x, $W^{xn\beta} S^{y\alpha} M \in G(mn)$ and the proof of the lemma is complete.

Since x and y were determined only modulo m it is clear that $(G(n): G(mn)) < \infty$.

On the basis of this lemma and formula (1.1), the theorem that follows can be proved in just the same way that the corresponding theorem for Γ was proved in [8].

THEOREM 1.4. *Let m and n be positive integers, and assume that $(m, \alpha\beta) = 1$. If H is any group such that*

$$G(mn) \subset H \subset G(n),$$

then $H = G(dn)$ for some divisor d of m.

This theorem can be generalized to a larger class of groups $G(m, n)$ defined as follows: The element

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of G belongs to $G(m, n)$ if and only if $b \equiv 0 \pmod{m}$, $c \equiv 0 \pmod{n}$. Thus $G(n) = G(1, n)$. We have as a matter of fact

THEOREM 1.5. *Let m and n be positive integers and assume that $(m, n\alpha\beta) = 1$. Let H be any subgroup of G containing $G(m, n)$. Then $H = G(m_1, n_1)$ for some divisors m_1 of m and n_1 of n.*

Proof. Since $(m, n\alpha\beta) = 1$, we can determine integers x, y such that $xm - yn\alpha\beta = 1$. Set

$$R = \begin{bmatrix} xm & \alpha \\ yn\beta & 1 \end{bmatrix}.$$

Then $R \in G$ since $R = S^{\alpha} W^{yn\beta}$. The proof is now completed in the same way as the proof of the corersponding theorem given in [9] for Γ.

For purposes of constructing modular functions we need more detailed information on the structure of the representatives of $G(n)$ modulo $G(mn)$ than is furnished by Lemma 1.1. This is given in the lemmas below:

LEMMA 1.2. *Let m and n be positive integers such that m is divisible only by primes dividing n and $(n, \beta) = 1$. Then $(G(n): G(mn)) = m$, and*

$$W^{xn\beta}, \qquad 0 \leqq x \leqq m - 1, \tag{1.3}$$

is a set of left coset representatives for $G(n)$ modulo $G(mn)$.

Proof. Let

$$M = \begin{bmatrix} a & b \\ nc & d \end{bmatrix} \in G(n).$$

Consider

$$W^{-x n\beta} M = \begin{bmatrix} * & * \\ n(c - xa\beta) & * \end{bmatrix}.$$

Now $(a\beta, n) = 1$ since $\det M = 1$ and $(\beta, n) = 1$ by assumption. Thus $(a\beta, m) = 1$ also, since m is divisible only by primes dividing n. Hence we can find x such that $xa\beta \equiv c \pmod m$, $0 \leqq x \leqq m-1$. For this x, $M = W^{xn\beta} M_0$, where $M_0 \in G(mn)$. Further, if $0 \leqq x, y \leqq m-1$, then $W^{-xn\beta} W^{yn\beta} \in G(mn)$ if and only if $n\beta(y-x) \equiv 0 \pmod{mn}$, which implies that $x = y$ since $(\beta, m) = 1$. The representatives (1.3) are thus distinct modulo $G(mn)$ and the proof of the lemma is completed.

LEMMA 1.3. *Let m and n be positive integers such that m is a power of a prime p and $(p, n\alpha\beta) = 1$. Set $m' = m/p$. Choose t so that $n\alpha\beta t \equiv -1 \pmod p$ and put $R = S^{\alpha} W^{nt\beta}$. Then $(G(n) : G(mn)) = m + m'$ and*

$$W^{xn\beta}, \quad 0 \leqq x \leqq m-1, \quad RW^{ynp\beta}, \quad 0 \leqq y \leqq m'-1, \tag{1.4}$$

is a set of left coset representatives for $G(n)$ modulo $G(mn)$.

Proof. Let

$$M = \begin{bmatrix} a & b \\ nc & d \end{bmatrix} \in G(n).$$

Suppose first that $(a, p) = 1$. Consider

$$W^{-xn\beta} M = \begin{bmatrix} * & * \\ n(c - xa\beta) & * \end{bmatrix}.$$

Since $(a\beta, p) = 1$ and m is a power of p, we can find x so that $xa\beta \equiv c \pmod m$, $0 \leqq x \leqq m-1$. For this x, $M = W^{xn\beta} M_0$, where $M_0 \in G(mn)$. Suppose now that $p \mid a$. Consider

$$W^{-xn\beta} S^{-\alpha} M = \begin{bmatrix} * & * \\ n(c - x\beta(a - \alpha nc)) & * \end{bmatrix}.$$

Now $(\beta(a - \alpha nc), p) = (\alpha\beta nc, p) = 1$ since $p \mid a$, $\det M = 1$, and $(n\alpha\beta, p) = 1$ by assumption. Thus x can be determined modulo m so that

$$x\beta(a - \alpha nc) \equiv c \pmod m. \tag{1.5}$$

For this x, $M = S^{\alpha} W^{xn\beta} M_0$, where $M_0 \in G(mn)$. Considering congruence (1.5) modulo p, we have $n\alpha\beta x \equiv -1 \pmod p$. Let t satisfy $n\alpha\beta t \equiv -1$

$(\mathrm{mod}\, p)$. Then $x = t + yp$, where now y is determined modulo m', and $S^\alpha W^{xn\beta}$ becomes $RW^{ynp\beta}$.

Further, suppose that $0 \leqq x, x' \leqq m-1$ and $0 \leqq y, y' \leqq m'-1$. Then $W^{-xn\beta}W^{x'n\beta} \in G(mn)$ if and only if $n\beta(x'-x) \equiv 0 \pmod{mn}$, or $x = x'$; $(RW^{ynp\beta})^{-1}RW^{y'np\beta} \in G(mn)$ if and only if $np\beta(y'-y) \equiv 0 \pmod{mn}$, or $y = y'$; and finally, $W^{-xn\beta}RW^{ynp\beta} \in G(mn)$ implies that $t \equiv 0 \pmod{p}$, which is impossible. The representatives (1.4) are therefore distinct modulo $G(mn)$, completing the proof of the lemma.

The application to the construction of modular functions we wish to make depends on these lemmas and Theorem 2.2 of [7]. We obtain

THEOREM 1.6. *Let m be a power of a prime p, and set $m' = m/p$. Suppose that $h(\tau)$ is a function on $G(m)$. Define*

$$f_m(\tau) = \sum_{k=0}^{m-1} h(W^{-k\beta}\tau),$$

$$g_m(\tau) = \sum_{k=0}^{m-1} h(W^{-kp\beta}\tau).$$

Then if $(p, \beta) = 1$, $g_m(\tau)$ is a function on $G(p)$ and if $(p, \alpha\beta) = 1$, $f_m(\tau) + g_{m'}(R^{-1}\tau)$ is a function on G, where $R = S^\alpha W^{t\beta}$ and $\alpha\beta t \equiv -1 \pmod{p}$.

Since R is in G, we can equally well conclude that in the latter case $f_m(R\tau) + g_{m'}(\tau)$ is a function on G, and since $g_{m'}(\tau)$ is a function on $G(p)$, $f_m(R\tau)$ is a function on $G(p)$ also.

Define the subgroup

$$K = [T, S^2].$$

Then K is of type $(2, 2)$ and of index 3 in Γ. In general $[T, S^n]$ is of type (n, n) and of infinite index for $n \geqq 3$. For $n = 1$ it is just Γ. K can also be described as follows: The element A of Γ is in K if and only if $A \equiv I \pmod{2}$ or $A \equiv T \pmod{2}$. From Lemmas 1.2 and 1.3 we obtain for K

LEMMA 1.4. *Let p be an odd prime, m a power of p, and set $m' = m/p$. Then $(K(p) : K(m)) = m'$ and*

$$W^{2pk}, \qquad 0 \leqq k \leqq m'-1, \tag{1.6}$$

is a set of left coset representatives for $K(p)$ modulo $K(m)$.

LEMMA 1.5. *Let p be an odd prime, m a power of p, and set $m' = m/p$. Put $R = S^2W^{\frac{1}{2}(p^2-1)}$. Then $(K : K(m)) = m + m'$ and*

(1.7) $$W^{2k}, \quad 0 \leqq k \leqq m-1, \quad RW^{2pk}, \quad 0 \leqq k \leqq m'-1,$$

is a set of left coset representatives for K modulo $K(m)$.

The group K is the underlying group for the study of the modular form whose powers generate $r_s(n)$. This study will be undertaken in § 3.

Let G be an arbitrary subgroup of Γ and let C be a fixed element of G, n a positive integer. Define the congruence subgroup $G(n;C)$ as follows: The element A of G belongs to $G(n;C)$ if and only if for same integer k, $A \equiv C^k \pmod{n}$.

These groups are congruence subgroups and have an easy structure. Thus $G(n;I)$ is the principal congruence subgroup of G of level n, and if e is the exponent of C modulo n, then $(G(n;C) : G(n;I)) = e$, the quotient group $G(n;C)/G(n;I)$ being isomorphic to the cyclic group generated by C modulo n. We remark that a principal congruence subgroup of G is normal in any subgroup of G containing it, and a subgroup of G containing a principal congruence subgroup is itself a congruence subgroup of G.

Let L be the complex of elements $A \in G(n;C)$ such that $A \equiv C \pmod{n}$. The following lemma is of use in showing a function invariant for $G(n;C)$:

LEMMA 1.6. *If* $g(\tau)$ *is invariant for* L, *then* $g(\tau)$ *is a function on* $G(n;C)$.

Proof. Since $C \in L$, $g(C\tau) = g(\tau)$; and by iteration, $g(\tau)$ is invariant for any power of C. Let $A \in G(n;C)$. Then for some integer k, $A \equiv C^k \pmod{n}$. Hence $C^{1-k}A \in L$ and so $g(\tau)$ is invariant for $C^{1-k}A$. Since $g(\tau)$ is invariant for C^{1-k}, $g(\tau)$ is invariant for A. The lemma is thus proved.

We note the facts

$$K = \Gamma(2;T), \qquad K(n) = \Gamma^*(n)(2;T), \ n \text{ odd}.$$

2. The modular form $\vartheta(\tau)$. The functions to be considered in the following section will be built up from the modular form $\vartheta(\tau)$, defined by

(2.1) $$\vartheta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = \exp \pi i \tau, \ \operatorname{im} \tau > 0.$$

Thus

$$\vartheta^s(\tau) = \sum_{n=0}^{\infty} r_s(n) q^n.$$

Here q is the proper uniformizing variable for $\vartheta(\tau)$ at the parabolic point $i\infty$ of the fundamental region of K, which will be denoted by K^*. A good discussion of uniformizing variables can be found in [2].

$\vartheta(\tau)$ can also be defined in terms of the Dedekind modular form $\eta(\tau)$, where

$$\eta(\tau) = x^{1/24}\prod_{n=1}^{\infty}(1-x^n), \quad x = q^2 = \exp 2\pi i\tau, \operatorname{im}\tau > 0, \tag{2.2}$$

and x is the proper uniformizing variable for $\eta(\tau)$ at the parabolic point $i\infty$ of Γ^*. The relationship is

$$\vartheta(\tau) = \eta^2(\tfrac{1}{2}(\tau+1))\eta^{-1}(\tau+1) = \prod_{n=1}^{\infty}(1-(-q)^n)(1+(-q)^n)^{-1}. \tag{2.3}$$

Both $\vartheta(\tau)$ and $\eta(\tau)$ are of dimension $-\frac{1}{2}$ and are regular and zero-free in the interior of the upper τ half-plane. Thus $\vartheta(\tau)$ is an entire modular form for K and $\eta(\tau)$ is an entire modular form for Γ.

The transformation equations for these functions are classical and have been worked out in great detail by many writers. In particular,

$$\vartheta(S^2\tau) = \vartheta(\tau), \tag{2.4}$$

$$\vartheta(T\tau) = (-i\tau)^{\frac{1}{2}}\vartheta(\tau) \tag{2.5}$$

and

$$\vartheta(W\tau) = 2(\tau+1)^{\frac{1}{2}}x^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-x^{2n})(1+x^n). \tag{2.6}$$

As (2.6) indicates, x is the proper uniformizing variable for $\vartheta(\tau)$ at the parabolic point 1 of K^*, since W takes $i\infty$ into 1.

In his paper [4] Hecke proves the transformation formula below, which we state as a lemma:

LEMMA 2.1. (*Hecke*, [4]). *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K, A \equiv T \pmod 2.$$

Then

$$\vartheta(A\tau) = (-i(c\tau+d))^{\frac{1}{2}}i^{\frac{1}{2}(1-c)}(a/|c|)\vartheta(\tau), \tag{2.7}$$

where $(a/|c|)$ *is the generalized Legendre-Jacobi symbol of quadratic reciprocity.*

To obtain the transformation formula for a matrix $A \in K$ such that $A \equiv I \pmod 2$, we write $A = T^{-1}TA$ and apply (2.5) and (2.7). In this manner we obtain

LEMMA 2.2. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K, A \equiv I \pmod 2.$$

Then

$$\vartheta(A\tau) = (-i(c\tau+d))^{\frac{1}{2}}i^{\frac{1}{2}(1-a)}(-c/|a|)\vartheta(\tau). \tag{2.8}$$

3. Sums of squares. We assume in this section that p is an odd prime and that m is an even power of p. We set $m' = m/p$. We prove first

THEOREM 3.1. *Let s be an integer, and define*

$$h(\tau) = h(s,p,m;\tau) = \vartheta^s(m\tau)\vartheta^{-s}(\tau). \tag{3.1}$$

Then $h(\tau)$ is an entire modular function on $K(m)$.

Proof. Suppose

$$A = \begin{bmatrix} a & b \\ mc & d \end{bmatrix} \in K(m).$$

Put

$$A_0 = \begin{bmatrix} a & mb \\ c & d \end{bmatrix}.$$

Then $A_0 \in K$, $A \equiv A_0 \pmod 2$, and we have

$$h(A\tau) = \vartheta^s(A_0 m\tau)\vartheta^{-s}(A\tau).$$

Since $K(m) = \Gamma_0(m)(2;T)$, we need only consider $A \equiv T \pmod 2$, in view of Lemma 1.6. Then Lemma 2.1 implies that

$$h(A\tau) = i^{\frac{1}{2}(m-1)cs}(a/m)^s h(\tau),$$

and this becomes $h(\tau)$ since m is an odd square. Thus $h(\tau)$ is a function on $K(m)$. The fact that $h(\tau)$ is an *entire* function on $K(m)$ is clear, since $\vartheta(\tau)$ is regular and zero-free in the interior of the upper τ half-plane.

We now want to construct a class of entire modular functions on K and we do this by Theorem 1.6, Lemmas 1.4 and 1.5 and the preceding theorem. we obtain

THEOREM 3.2. *Put*

$$f_m(\tau) = f(s,p,m;\tau) = \sum_{k=0}^{m-1} h(W^{-2k}R\tau), \tag{3.2}$$

$$g_m(\tau) = g(s,p,m;\tau) = \sum_{k=0}^{m-1} h(W^{-2pk}\tau), \tag{3.3}$$

where $h(\tau)$ is defined by (3.1) and $R = S^2W^{\frac{1}{2}(p^2-1)}$. Then $f_m(\tau)$, $g_{m'}(\tau)$, $g_m(\tau)$ are entire modular functions on $K(p)$ and

$$F_m(\tau) = f_m(\tau) + g_{m'}(\tau) \tag{3.4}$$

is an entire modular function on K.

Our general procedure will be as follows: K^*, which is of genus zero, has the two inequivalent parabolic points $\tau = i\infty$, $\tau = 1$. The functions $F_m(\tau)$ are entire modular functions on K with polar singularities at most in the proper uniformizing variables (q for $i\infty$, x for 1) at these parabolic points. Therefore we need only determine the behaviour of these functions at $i\infty$ and 1 to have complete information about them. We shall show that $F_m(\tau)$ is pole-free at $\tau = i\infty$, and for positive s has a pole at $\tau = 1$ of order less than $s/8$, a number *independent of* m. This will imply that for sufficiently many different values of m the functions $F_m(\tau)$ are linearly dependent, if $s > 0$. Comparing coefficients we will obtain the desired identities for $r_s(n)$.

We consider $\tau = i\infty$ first. The function $f_m(\tau)$ can be simplified by noticing that

$$R^{-1}T = \begin{bmatrix} -2 & -1 \\ p^2 & \frac{1}{2}(p^2-1) \end{bmatrix} \in K(p);$$

and since $f_m(\tau)$ is a function on $K(p)$, we have

$$\begin{aligned} f_m(\tau) &= f_m(R^{-1}T\tau) \\ &= \sum_{k=0}^{m-1} h(W^{-2k}T\tau) \\ &= \sum_{k=0}^{m-1} h(TS^{2k}\tau) \\ &= \sum_{k=0}^{m-1} \vartheta^s(T(\tau+2k)/m)\vartheta^{-s}(T(\tau+2k)). \end{aligned}$$

The transformation formula (2.5) may now be employed to show that

$$f_m(\tau) = m^{-\frac{1}{2}s}\vartheta^{-s}(\tau)\sum_{k=0}^{m-1}\vartheta^s((\tau+2k)/m). \tag{3.5}$$

Therefore $f_m(\tau)$ is pole-free at $\tau = i\infty$ since $\vartheta(\tau)$ is both pole and zero-free at $\tau = i\infty$.

We now consider $g_{m'}(\tau)$ at $\tau = i\infty$. We have

$$\begin{aligned} g_{m'}(\tau) &= \sum_{k=0}^{m'-1} h(W^{-2pk}\tau) \\ &= \sum_{k=0}^{m'-1} \vartheta^s(mW^{-2pk}\tau)\vartheta^{-s}(W^{-2pk}\tau) \\ &= \sum_{k=0}^{m'-1} \vartheta^s(m'W^{-2k}p\tau)\vartheta^{-s}(W^{-2pk}\tau). \end{aligned}$$

Let $(m', k) = \Delta$ and put $m' = \Delta m_0$, $k = \Delta k_0$, where now $(m_0, k_0) = 1$. Determine integers m_1, k_1 so that $m_0m_1 - 4k_0k_1 = 1$ and set

$$M = \begin{bmatrix} m_0 & -2k_1 \\ -2k_0 & m_1 \end{bmatrix}.$$

Then $M \in K$ and $M \equiv I \pmod 2$. Furthermore,

$$m'W^{-2k}p\tau = M(\Delta p\tau + 2k_1)/(m'/\Delta).$$

Now $W^{-2pk} \in K$, $W^{-2pk} \equiv I \pmod 2$ also. Thus using (2.8) we find that

$$h(W^{-2pk}\tau) = c\vartheta^s((\Delta p\tau + 2k)/(m'/\Delta))\vartheta^{-s}(\tau),$$

where c is a number independent of τ. Therefore $g_{m'}(\tau)$ is also pole-free at $\tau = i\infty$ and we have proved

LEMMA 3.1. *For all integral s, $F_m(\tau)$ is pole-free at $\tau = i\infty$.*

We go on now to the remaining parabolic point $\tau = 1$. As indicated, we study the function $F_m(W\tau)$ at $\tau = i\infty$ in terms of the uniformizing variable $x = \exp 2\pi i\tau$. As with $\tau = i\infty$, we consider each component $f_m(W\tau)$, $g_{m'}(W\tau)$ separately. Now

$$g_{m'}(W\tau) = \sum_{k=0}^{m'-1} h(W^{-2pk+1}\tau)$$
$$= \sum_{k=0}^{m'-1} \vartheta^s(mW^{-2pk+1}\tau)\vartheta^{-s}(W^{-2pk+1}\tau).$$

But $(m, -2pk+1) = 1$ since m is power of p. Thus integers m_1, k_1 can be determined so that $mm_1 - 2(2pk-1)k_1 = 1$. Put

$$M = \begin{bmatrix} m & -2k_1 \\ 1-2pk & m_1 \end{bmatrix}.$$

Then

$$mW^{-2pk+1}\tau = M(\tau + 2k_1)/m.$$

Now $MW^{-1} \in K$, and $MW^{-1} \equiv I \pmod 2$. Using (2.8) once again, we find that for some number c independent of τ,

$$h(W^{-2pk+1}\tau) = c\vartheta^s(W(\tau + 2k_1)/m)\vartheta^{-s}(W\tau).$$

We can now use (2.6) to deduce that apart from constant multipliers, the latter expression begins with

$$x^{-\epsilon}, \epsilon = (s/8)(1 - 1/m).$$

From this we conclude that $g_{m'}(W\tau)$ is pole-free if $s \leqq 0$ and has a pole of order less than $s/8$ if $s > 0$.

Turning now to $f_m(W\tau)$ and making use of (3.5) we have that

$$f_m(W_\tau) = m^{-\frac{1}{2}s}\vartheta^{-s}(W_\tau)\sum_{k=0}^{m-1}\vartheta^s((1/m)S^{2k}W_\tau).$$

Put $\Delta = (2k+1, m)$. Set $2k+1 = \Delta k_0, m = \Delta m_0$ so that $(m_0, k_0) = 1$. Determine k_1, m_1 so that $2k_0k_1 - m_0m_1 = 1$. Put

$$M = \begin{bmatrix} k_0 & m_1 \\ m_0 & 2k_1 \end{bmatrix}.$$

Then

$$(1/m)S^{2k}W_\tau = M(\Delta\tau + \Delta - 2k_1)/(m/\Delta).$$

Furthermore, $MW^{-1} \in K$ and $MW^{-1} \equiv T \pmod 2$. Using (2.7) and then (2.6), we find that

$$\vartheta^s((1/m)S^{2k}W_\tau)\vartheta^{-s}(W_\tau)$$

begins with

$$x^{-\epsilon}, \epsilon = (s/8)(1 - \Delta^2/m),$$

apart from constant multipliers. Considering $s \leqq 0$ and $s > 0$ separately, we conclude easily

LEMMA 3.2. *If $s \leqq 0$, then $F_m(\tau)$ has a pole of order $-s(m^2-1)/8$ at most at $\tau = 1$. If $s > 0$, then $F_m(\tau)$ has a pole of order less than $s/8$ at $\tau = 1$.*

Summarizing for $s > 0$, Lemmas 3.1 and 3.2 taken together state

THEOREM 3.3. *Suppose $s > 0$. Then the total valence of $F_m(\tau)$ throughout K^* is less than $s/8$.*

From this theorem we deduce the following one by a simple application of Liouville's theorem:

THEOREM 3.4. (*First principal theorem*). *Let μ be any integer satisfying*

$$\mu \geqq s/8, \qquad s > 0.$$

Then if m_k, $1 \leqq k \leqq \mu$, are all even powers of p there exist constants C_k, $0 \leqq k \leqq \mu$, not all zero such that

$$\sum_{k=1}^{\mu} C_k F_{m_k}(\tau) = C_0. \tag{3.6}$$

Identity (3.6) is the identity whose existence we wished to prove. We are now interested is comparing coefficients in this identity to obtain identies for $r_s(n)$, but first we must know the Fourier expansion of $F_m(\tau)$ at $\tau = i\infty$.

The component $f_m(\tau)$ offers no difficulty whatsoever and we find

$$f_m(\tau) = m^{1-\frac{1}{2}s}\vartheta^{-s}(\tau)\sum_{n=0}^{\infty} r_s(mn)q^n. \tag{3.7}$$

$g_{m'}(\tau)$ is quite complicated however and we must resort to special devices.

We write $k:n$ in a sumation to indicate that k runs over a reduced set of residues modulo n. Then if ϕ is an arbitrary function and n is a power of p, it is evident that

$$\sum_{k=0}^{n-1}\phi(k) = \sum_{k:n}\phi(k) + \sum_{k=0}^{n'-1}\phi(pk), \tag{3.8}$$

where $n' = n/p$. Iterating, we can prove without difficulty

LEMMA 3.3. *If $n = p^t$, then*

$$\sum_{k=0}^{n-1}\phi(k) = \phi(0) + \sum_{r=1}^{t}\sum_{k:p^r}\phi(kp^{t-r}). \tag{3.9}$$

If we apply Lemma 3.3 to $g_{m'}(\tau)$, we find

LEMMA 3.4. *Suppose $m = p^{2t}$. Then*

$$g_{m'}(\tau) = h(\tau) + \sum_{r=1}^{2t-1}\phi_r(\tau), \tag{3.10}$$

where

$$\phi_r(\tau) = \sum_{k:p^r} h(W^{-2p^{2t-r}k}\tau). \tag{3.11}$$

In this form we can expand $g_{m'}(\tau)$. We have

$$\begin{aligned} h(W^{-2p^{2t-r}k}\tau) &= \vartheta^s(p^{2t}W^{-2p^{2t-r}k}\tau)\,\vartheta^{-s}(W^{-2p^{2t-r}k}\tau) \\ &= \vartheta^s(p^rW^{-2k}p^{2t-r}\tau)\,\vartheta^{-s}(W^{-2p^{2t-r}k}\tau). \end{aligned}$$

Now $(p^r, k) = 1$ since k runs over a reduced set of residues modulo p^r. Thus integers m_1, k_1 can be determined so that $p^rm_1 - 4kk_1 = 1$. Put

$$M = \begin{bmatrix} p^r & -2k_1 \\ -2k & m_1 \end{bmatrix}.$$

Then $M \in K$, $M \equiv I \pmod 2$. Also

$$p^rW^{-2k}p^{2t-r}\tau = M(p^{2t-r}\tau + 2k_1)/p^r.$$

We can now employ (2.8) to simplify $h(W^{-2p^{2t-r}k}\tau)$. The result is that

$$h(W^{-2p^{2t-r}k}\tau) = c\vartheta^s((p^{2t-r}\tau + 2k_1)/p^r)\,\vartheta^{-s}(\tau),$$

where

$$c = p^{-\frac{1}{2}rs} i^{\frac{1}{2}s(1-p^r)} (-2k_1/p)^{rs}.$$

Noting now that k and k_1 simultaneously run over a reduced set of residues modulo p^r, we find

$$g_{m'}(\tau) = h(\tau) + \vartheta^{-s}(\tau) \sum_{r=1}^{2t-1} p^{-\frac{1}{2}rs} i^{\frac{1}{2}s(1-p^r)} (-2/p)^{rs} G_r(\tau),$$

where

$$G_r(\tau) = \sum_{k:p^r} (k/p)^{rs} \vartheta^s((p^{2t-r}\tau + 2k)/p^r).$$

The expansion of $G_r(\tau)$ is straightforward and depends only on the Gauss sums. We find that

$$G_r(\tau) = p^{r-1} \sum_{n=0}^{\infty} \sum_{k:p} (k/p)^{rs} e^{2\pi i nk/p} r_s(p^{r-1}n) q^{p^{2t-1-r}n}. \tag{3.12}$$

Let us define

$$\gamma_{rs}(n) = \begin{cases} \sum_{k:p} (k/p)^{rs} e^{2\pi i nk/p} & n \text{ an integer}, \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

so that for integral n

$$\gamma_{rs}(n) = \begin{cases} p-1 & rs \text{ even}, \ p \mid n \\ -1 & rs \text{ even}, \ (n,p) = 1 \\ i^{(\frac{1}{4}(p-1))^2} (n/p) p^{\frac{1}{2}} & rs \text{ odd}. \end{cases}$$

Then (3.12) may be written

$$G_r(\tau) = p^{r-1} \sum_{n=0}^{\infty} \gamma_{rs}(p^{r+1-2t}n) r_s(p^{2r-2t}n) q^n.$$

We thus obtain

LEMMA 3.5. *Define*

$$C_{r,s} = p^{r-\frac{1}{2}rs-1} i^{\frac{1}{2}s(1-p^r)} (-2/p)^{rs}.$$

Then

$$g_{m'}(\tau) = \vartheta^{-s}(\tau) \sum_{n=0}^{\infty} \{r_s(n/m) + \sum_{r=1}^{2t-1} C_{r,s}\gamma_{rs}(p^{r+1-2t}n) r_s(p^{2r-2t}n)\} q^n. \tag{3.14}$$

Expansion (3.14) now yields our second principal theorem.

THEOREM 3.5. (*Second principal theorem*). *Let μ be any integer satisfying*

$$\mu \geqq s/8, \quad s > 0.$$

choose $m_k = p^{2t_k}$, $1 \leqq k \leqq \mu$. *Then there are numbers* c_k, $0 \leqq k \leqq \mu$, *not all zero and independent of* n *such that for all integral* n

$$\sum_{k=1}^{\mu} C_k\{m_k^{1-\frac{1}{2}s} r_s(m_k n) + r_s(n/m_k) + \sum_{r=1}^{2t_k-1} C_{r,s\gamma rs}(p^{r+1-2t_k}n) r_s(p^{2r-2t_k}n)\} = C_0 r_s(n). \tag{3.15}$$

The simplest examples (and the only ones we shall write down explicitly) occur for $s \leqq 8$. For then the choice $\mu = 1$ is permissible, and choosing $m = p^2$, we find

$$r_s(np^2) = \{1 + p^{s-2} - (-1)^{\frac{1}{2}(s-1)(p-1)} p^{\frac{1}{2}(s-3)}(n/p)\} r_s(n) - p^{s-2} r_s(n/p^2), \qquad s = 1, 3, 5, 7, \tag{3.16}$$

$$r_s(np^2) = \{1 + p^{s-2} + (-1)^{\frac{1}{2}s(p-1)} p^{\frac{1}{2}(s-2)}(n/p)^2\} r_s(n) - p^{s-2} r_s(n/p^2), \qquad s = 2, 4, 6, 8. \tag{3.17}$$

The identity for $s = 3$ can be found in [11]. See also the paper [5] where the identities for $s = 3, 5, 7$ are derived by the application of a Hecke operator.

As a final application, we notice that the entire modular function $F(-1, 3, 9; \tau)$ is non-constant, and is regular throughout K^* except at $\tau = 1$ where it has a pole of order 1. Therefore

THEOREM 3.6. *The function* $F(-1, 3, 9; \tau)$ *is a Hauptmodul for* K.

NATIONAL BUREAU OF STANDARDS,
WASHINGTON, D. C.

REFERENCES.

[1] J. L. Brenner, "Quelques groupes libres de matrices," *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, vol. 241 (1955), pp. 1689-1691.

[2] L. R. Ford, *Automorphic Functions*, New York, 1951.

[3] K. Goldberg and M. Newman, "Pairs of matrices of order two which generate free groups," *Illinois Journal of Mathematics*, vol. 1 (1957), pp. 446-448.

[4] E. Hecke, "Herleitung des Euler-Produktes der Zetafunktion und einiger L-reihen aus ihrer Funktionalgleichung," *Mathematische Annalen*, vol. 119 (1944), pp. 266-287.

[5] J. H. Lint, *Hecke operators and Euler products*, Dissertation, University of Utrecht, 1957.

[6] M. Newman, "On the existence of identities for the coefficients of certain modular forms," *The Journal of the London Mathematical Society*, vol. 30 (1955), pp. 488-493.

[7] ———, "Remarks on some modular identities," *Transactions of the Mathematical Society*, vol. 73 (1952), pp. 313-320.

[8] ———, "Structure theorems for modular subgroups," *Duke Mathematical Journal*, vol. 22 (1955), pp. 25-32.

[9] ———, "An inclusion theorem for modular groups," *Proceedings of the American Mathematical Society*, vol. 8 (1957), pp. 125-127.

[10] ———, "Modular forms whose coefficients possess multiplicative properties," *Annals of Mathematics*, vol. 70 (1959), pp. 478-489.

[11] G. Pall, "On the arithmetic of quaternions," *Transactions of the American Mathematical Society*, vol. 47 (1940), pp. 487-500.

ABELIAN EXTENSIONS OF DIFFERENTIAL FIELDS.*

By E. R. KOLCHIN.[1]

Introduction. The Galois theory of differential fields ([2], [3]) is concerned with strongly normal extensions of a differential field $\mathfrak{F}$ with algebraically closed field of constants $\mathfrak{C}$ of characteristic 0. The Galois group $\mathfrak{G}$ of such an extension $\mathfrak{G}$ has a sort of algebraico-geometric structure and is shown to be birationally isomorphic with an algebraic group variety (not necessarily connected); more precisely, there is an algebraic group G in the algebraic geometry having for universal domain the fixed universal differential field extension $\mathfrak{U}$ of $\mathfrak{G}$, G being defined over $\mathfrak{C}$, such that $\mathfrak{G}$ is birationally isomorphic with the group $G_{\mathfrak{K}}$ consisting of the points of G which are rational over the field of constants $\mathfrak{K}$ of $\mathfrak{U}$ ($G_{\mathfrak{K}}$ is an algebraic group in the algebraic geometry with universal domain $\mathfrak{K}$). This permits the classification of strongly normal extensions by means of algebraic groups: a strongly normal extension of $\mathfrak{F}$ with Galois group birationally isomorphic to the subgroup $G_{\mathfrak{K}}$ of an algebraic group G defined over $\mathfrak{C}$, or to an algebraic subgroup of $G_{\mathfrak{K}}$, will be called a *G-extension* of $\mathfrak{F}$.

If $\mathfrak{G}$ is a G-extension of $\mathfrak{F}$ and if

$$G = G_0 \supset G_1 \supset \cdots \supset G_r = 1$$

is a normal chain of algebraic subgroups of G, all defined over $\mathfrak{C}$, then there is a corresponding tower

$$\mathfrak{F} = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_r = \mathfrak{G}$$

of differential fields such that $\mathfrak{F}_i$ is a G_{i-1}/G_i-extension of $\mathfrak{F}_{i-1}$ $(1 \leqq i \leqq r)$. Now, by Chevalley's structure theorem (see Rosenlicht [4], p. 439, or Barsotti [1], p. 47) G has a canonical normal chain

$$G \supset G^0 \supset H^0 \supset 1 \tag{1}$$

defined over $\mathfrak{C}$, G^0 being the component of the unity element of G and H^0 being the greatest connected linear algebraic subgroup of G^0, such that the

---

* Received October 26, 1959.

[1] This work was done in connection with a contract with the Office of Naval Research.

factor groups G/G^0 and G^0/H^0 are, respectively, finite and abelian (i.e. an abelian variety). Therefore $\mathfrak{F}$ and $\mathfrak{G}$ appear in a corresponding tower

$$\mathfrak{F} \subset \mathfrak{F}^0 \subset \mathfrak{E}^0 \subset \mathfrak{G}. \tag{2}$$

But a finite group corresponds to an extension of finite degree, so that $\mathfrak{F}^0$ is finite algebraic over $\mathfrak{F}$ (indeed, $\mathfrak{F}^0$ is the algebraic closure of $\mathfrak{F}$ in $\mathfrak{G}$), and ([3] p. 891) a linear group corresponds to a Picard-Vessiot extension, so that $\mathfrak{G}$ is Picard-Vessiot over $\mathfrak{E}^0$. Thus, the gap in our knowledge of the structure of $\mathfrak{G}$ over $\mathfrak{F}$ is in the extension from $\mathfrak{F}^0$ to $\mathfrak{E}^0$. The purpose of the present paper is to close this gap, that is, to characterize the strongly normal extensions which are A-extensions for some abelian variety A. We shall call any such extension an *abelian* extension. The characterization which we find is in terms of solutions of certain kinds of differential equations, and generalizes the result previously obtained ([2] p. 809, [3] p. 892) in the case in which the extension is of transcendence degree 1, i.e. A is of dimension 1. The remainder of the present introduction is intended to describe this characterization more fully, in its natural more general setting which includes, among others, the above mentioned characterization of strongly normal extensions with linear groups.

Let G be any *connected* algebraic group defined over $\mathfrak{C}$, of dimension say n. If ω is a differential on G and α is a point of G at which ω is finite, there is induced a differential $\omega(\alpha)$ of $\mathfrak{U}$ over $\mathfrak{K}$ (if $\omega = df$ and the rational function f on G is defined over $\mathfrak{K}$ then $\omega(\alpha)$ maps each derivation $\mathfrak{d}$ of $\mathfrak{U}$ over $\mathfrak{K}$ onto $\mathfrak{d}f(\alpha)$). Every birational automorphism ϕ of G induces a contravariant automorphism ϕ^* of the vector space $\mathfrak{Z}(G)$ over $\mathfrak{U}$ of *invariant* (i.e. left invariant) differentials on G; in particular, the inner automorphism $\tau(\beta)$: $s \to \beta s \beta^{-1}$ determined by an element β of G determines an automorphism $\tau(\beta)^*$ of $\mathfrak{Z}(G)$. It is "well-known" that if ω is invariant, and therefore finite at every point of G, then

$$\omega(\alpha\beta) = (\tau(\beta^{-1})^*\omega)(\alpha) + \omega(\beta) \tag{3}$$

for all points α and β of G. This formula (which when G is the additive or multiplicative group of $\mathfrak{U}$ becomes, respectively, $\mathfrak{d}(\alpha+\beta) = \mathfrak{d}\alpha + \mathfrak{d}\beta$ or $(\alpha\beta)^{-1}\mathfrak{d}(\alpha\beta) = \alpha^{-1}\mathfrak{d}\alpha + \beta^{-1}\mathfrak{d}\beta$, $\mathfrak{d}$ representing an arbitrary derivation of $\mathfrak{U}$ over $\mathfrak{K}$, and when G is the general linear group $GL(r)$ becomes $(\alpha\beta)^{-1}\mathfrak{d}(\alpha\beta) = \beta^{-1}(\alpha^{-1}\mathfrak{d}\alpha)\beta + \beta^{-1}\mathfrak{d}\beta$) has been given a rigorous exposition by Rosenlicht [5], but unfortunately only in the case of commutative groups G; therefore we give in § 1 a discussion supplementing that of Rosenlicht, establishing the desired formula in full generality.

We define a point α of G to be a *G-primitive* over $\mathfrak{F}$ if the following two conditions are satisfied: 1° the field of constants of $\mathfrak{F}\langle\alpha\rangle$ is $\mathfrak{k}$; 2° $\langle\delta, \omega(\alpha)\rangle \in \mathfrak{F}$ for every derivation operator δ of the differential field and every invariant differential ω on G defined over $\mathfrak{k}$. If the derivation operators are denoted by $\delta_1, \cdots, \delta_m$ and if n linearly independent differentials $\omega_1, \cdots, \omega_n$ defined over $\mathfrak{k}$ are expressed in the form

$$\omega_j = \sum_{1 \leqq \nu \leqq n} f_{j\nu}\, dx_\nu \qquad (1 \leqq j \leqq n),$$

where the $f_{j\nu}$ and the x_ν are rational functions on G defined over $\mathfrak{k}$ and defined at α, and $x_1, \cdots, x_n$ are algebraically independent over $\mathfrak{k}$, then the condition 2° above is equivalent to the condition that α satisfy a system of differential equations

$$\sum_{1 \leqq \nu \leqq n} f_{j\nu}(\alpha)\, \delta_i x_\nu(\alpha) = a_{ij} \qquad (1 \leqq i \leqq m, 1 \leqq j \leqq n),$$

where the a_{ij} are elements of $\mathfrak{F}$.

By a *G-primitive extension* of $\mathfrak{F}$, we mean an extension of $\mathfrak{F}$ of the form $\mathfrak{F}\langle\alpha\rangle$, where α is a G-primitive over $\mathfrak{F}$.

It is not difficult to prove, making use of (3), that a necessary and sufficient condition for α to be a G-primitive over $\mathfrak{F}$ is that the field of constants of $\mathfrak{F}\langle\alpha\rangle$ be $\mathfrak{k}$ and $\sigma\alpha \cdot \alpha^{-1} \in G_{\mathfrak{K}}$ for every $\mathfrak{F}$-isomorphism σ of $\mathfrak{F}\langle\alpha\rangle$ onto an extension of $\mathfrak{F}$ in $\mathfrak{U}$. It follows almost immediately that every G-primitive extension of $\mathfrak{F}$ is a G-extension of $\mathfrak{F}$.

Suppose now that $\mathfrak{G}$ is a G-extension of $\mathfrak{F}$, so that there exists an injective rational homomorphism $\gamma: \mathfrak{G} \to G_{\mathfrak{K}}$, $\mathfrak{G}$ denoting the Galois group of $\mathfrak{G}$ over $\mathfrak{F}$. Regarding γ as a rational crossed homomorphism of $\mathfrak{G}$ into G, we find by the above that *if γ splits in $G_{\mathfrak{G}}$ (in particular if the cohomology set $H^1(\mathfrak{G}, G)$ is trivial) then $\mathfrak{G}$ is a G-primitive extension of $\mathfrak{F}$.* It is known ([3] p. 887) that $H^1(\mathfrak{G}, G)$ is trivial whenever G is the additive group of $\mathfrak{U}$, or the multiplicative group of $\mathfrak{U}$, or the general linear group $GL(r)$ for any r; this recaptures the characterizations ([2] p. 809 and [3] p. 891) of the strongly normal extensions corresponding to these groups as extensions of the form $\mathfrak{F}\langle\alpha\rangle$, where in the first case α is an element such that $\delta\alpha \in \mathfrak{F}$ for each derivation operator δ (extension by a *primitive* element), in the second case α is an element such that $\alpha^{-1}\delta\alpha \in \mathfrak{F}$ for each δ (extension by an *exponential* element), and in the third case α is an invertible $r \times r$ matrix such that $\alpha^{-1}\delta\alpha$ has its coordinates in $\mathfrak{F}$ for each δ (*Picard-Vessiot* extension).

If G is commutative, in particular if G is an abelian variety A, then we do not know in general that $H^1(\mathfrak{G}, G)$, which is now a group, is trivial; but we do know ([3] p. 887) that every element of $H^1(\mathfrak{G}, G)$ is of finite order,

so that for a suitable exponent $h > 0$ the cocycle γ^h splits in $G_{\mathcal{G}}$. It follows in this case that there is a G-primitive α such that $\mathcal{F} \subset \mathcal{F}\langle\alpha\rangle \subset \mathcal{G}$ and the Galois group of $\mathcal{G}$ over $\mathcal{F}\langle\alpha\rangle$ equals the finite group $\mathfrak{G}(h)$ of all elements $\sigma \in \mathfrak{G}$ with $\sigma^h = 1$.

Returning to the general case of a G-extension $\mathcal{G}$ with arbitrary algebraic group G defined over $\mathcal{C}$, we see that thse normal chain (1) has a refinement

$$G \supset G^0 \supset H \supset H^0 \supset 1,$$

where H is the set of points $s \in G^0$ such that $s^h \in H$, and therefore the group variety $A = G^0/H$ is abelian and H/H^0 is finite abelian, and that the tower (2) has a corresponding refinement

$$\mathcal{F} \subset \mathcal{F}^0 \subset \mathcal{E} \subset \mathcal{E}^0 \subset \mathcal{G},$$

where $\mathcal{E}$ is A-primitive over $\mathcal{F}^0$ and $\mathcal{E}^0$ is an abelian extension of $\mathcal{E}$ of finite degree. Since H/H^0 is finite and H^0 is linear, it easily follows that H is birationally isomorphic with a linear algebraic group, and therefore that $\mathcal{G}$ is a Picard-Vessiot extension of $\mathcal{E}$. Thus *every strongly normal extension has a three-storey tower*

$$\mathcal{F} \subset \mathcal{F}^0 \subset \mathcal{E} \subset \mathcal{G}$$

in which $\mathcal{F}^0$ (the algebraic closure of $\mathcal{F}$ in $\mathcal{G}$) is a normal extension of $\mathcal{F}$ of finite degree, $\mathcal{E}$ is an A-primitive extension of $\mathcal{F}^0$ for some abelian variety A defined over $\mathcal{C}$, and $\mathcal{G}$ is a Picard-Vessiot extension of $\mathcal{E}$.

In the final § 3 the A-primitive extensions are given a function-theoretic characterization in the classical case in which $\mathcal{F}$ consists of functions of m complex variables meromorphic in some region of $\boldsymbol{C}^m$. It is shown that if A is an abelian variety of dimension n defined over the field $\boldsymbol{C}$ of complex numbers then *every A-primitive extension of $\mathcal{F}$ is of the form $\mathcal{F}(f_1(\zeta), \cdots, f_r(\zeta))$, where $f_1(z), \cdots, f_r(z)$ are abelian functions of $z = (z_1, \cdots, z_n)$ which generate a nondegenerate abelian function field, and $\zeta = (\zeta_1, \cdots, \zeta_n)$ is a family of n functions of m complex variables, each ζ_j being primitive over $\mathcal{F}$.*

1. Induced differentials. This section supplements the discussion of Rosenlicht [5]; a familiarity with the terminology and notation as well as with the methods and results of that paper is assumed. We find it convenient, for any element u of a vector space and any element v of the dual space, to denote the value of v at u by $\langle u, v\rangle$.

Let V be an algebraic variety defined over a field k, let K be a subfield of the universal domain $\mathcal{U}$ (not necessarily distinct from $\mathcal{U}$) with $k \subset K$, and let $\mathfrak{D}(K/k)$ denote the vector space over K formed by the derivations

of K over k. We recall (Rosenlicht [5], p. 54) that if α is a simple point on V which is rational over K then there is a unique K-linear mapping $\mathfrak{d} \to \mathfrak{d}_\alpha$, from $\mathfrak{D}(K/k)$ to the space of tangent vectors to V at α which are rational over K, defined by the condition that $\mathfrak{d}_\alpha f = \mathfrak{d} f(\alpha)$ $(f \in \mathfrak{o}_\alpha \cap k(V))$; also, if ω is any differential on V which is finite at α and defined over K, then the local component ω_α of ω at α is a K-linear mapping of the tangent space to V at α into $\mathfrak{U}$ which maps the tangent vectors rational over K into K. The composite of the two mappings $\mathfrak{d} \to \mathfrak{d}_\alpha$ and ω_α then gives a K-linear mapping of $\mathfrak{D}(K/k)$ into K, that is, a differential of K over k; we call it the differential of K over k *induced by* ω *at* α, and following Rosenlicht denote it by $\omega(\alpha)$, or by $\omega_{K/k}(\alpha)$ when the greater detail is desirable.

If $f_i, g_i \in \mathfrak{o}_\alpha \cap k(V)$ $(1 \leqq i \leqq r)$ and $\omega = \sum f_i dg_i$ then

$$\langle \mathfrak{d}, \omega(\alpha) \rangle = \sum f_i(\alpha) \mathfrak{d} g_i(\alpha).$$

It is easy to see that if a subfield k_0 of k is also a field of definition of V and ω is defined over k_0 then $\langle \mathfrak{d}, \omega(\alpha) \rangle \in k_0(\alpha) \cdot k_0(\mathfrak{d} k_0(\alpha))$. Furthermore, if V' is also a variety defined over k, and $\phi: V \to V'$ is a rational mapping defined over k such that ϕ is defined at α and $\phi(\alpha)$ is a simple point on V', and ω' is a differential on V' finite at $\phi(\alpha)$ and defined over K, then $\omega'(\phi(\alpha)) = (\phi^* \omega')(\alpha)$.

We are interested in the case in which V is a group variety G and ω is an invariant (meaning left invariant) differential on G. Every point of G is simple, and ω is finite everywhere. In order to prove the result we require, we use the notion of a rational crossed homomorphism.

Let G and G' be connected algebraic groups. We say that G *operates on* G' *by* g if g is a group homomorphism of G into the group of automorphisms of G' such that the mapping $G \times G' \to G'$ defined by $(s, s') \to g(s)s'$ is rational. When such is the case $g(s)$ is a birational automorphism of G', defined over $k_0(s)$, whenever k_0 is a common field of definition of G, of G', and of the mapping $(s, s') \to g(s)s'$.

Suppose G operates on G' by g. A *rational crossed homomorphism* (or 1-*cocycle*) of G into G' (relative to g) is a rational mapping $\zeta: G \to G'$ such that, if k_0 is a common field of definition of G, of G', of the mapping $(s, s') \to g(s)s'$, and of ζ, and if (s, t) is a generic point of $G \times G$ over k_0, then $\zeta(st) = \zeta(s) \cdot g(s)\zeta(t)$. It is easy to see that this implies that ζ is defined at every point of G, $\zeta(st) = \zeta(s) \cdot g(s)\zeta(t)$ for all points s, t of G, and $\zeta(1) = 1$. (We denote the unity element of every multiplicative group by 1.) The notion of rational crossed homomorphism includes that of rational homomorphism, the latter corresponding to the case in which the operation g is trivial, that is $g(s)$ is the identity automorphism of G' for every $s \in G$.

Consider a rational crossed homomorphism $\zeta: G \to G'$. ζ induces a homomorphism $X \to \zeta X$ of the tangent space to G at 1 into the tangent space to G' at $1 = \zeta(1)$; since these tangent spaces are canonically isomorphic (as vector spaces) to the respective Lie algebras $\mathfrak{L}(G)$ and $\mathfrak{L}(G')$ of G and G', we obtain a linear mapping $D \to \zeta D$ of $\mathfrak{L}(G)$ into $\mathfrak{L}(G')$. Thus, by definition, $(\zeta D)_1 = \zeta D_1$ $(D \in L(G))$. (We use a point of a variety as a subscript to a derivation or differential on the variety to indicate the local component at that point.)

This being so for any rational crossed homomorphism, in particular it is so for any rational homomorphism, including the automorphism $g(s): G' \to G'$.

Returning to ζ, we note that the crossed homomorphism property $\zeta(st) = \zeta(s) \cdot g(s)\zeta(t)$ can be written in the form $\zeta T_s = T_{\zeta(s)} g(s) \zeta$, where for any element v of any group we use T_v to denote the left multiplication mapping $x \to vx$ of that group into itself. It follows that for any $D \in \mathfrak{L}(G)$ we may write $\zeta D_s = \zeta T_s D_1 = T_{\zeta(s)} g(s) \zeta D_1 = T_{\zeta(s)} (g(s)\zeta D)_1 = (g(s)\zeta D)_{\zeta(s)}$, so that

$$(g(s)\zeta D)_{\zeta(s)} = \zeta D_s \qquad (D \in \mathfrak{L}(G), s \in G).$$

Therefore the homomorphism $\omega' \to \zeta^* \omega'$ which ζ induces from the space of differentials on G' into the space of differentials on G has the property that $\langle D, \zeta^* \omega' \rangle(s) = \langle \zeta D_s, \omega'_{\zeta(s)} \rangle = \langle (g(s)\zeta D)_{\zeta(s)}, \omega'_{\zeta(s)} \rangle = \langle g(s)\zeta D, \omega' \rangle(\zeta(s))$. But if ω' is an element of the space $\mathfrak{I}(G')$ of *invariant* differentials on G' then $\langle g(s)\zeta D, \omega' \rangle(t')$ is independent of the element $t' \in G'$, so that we may write

$$(4) \qquad \langle D, \zeta^* \omega' \rangle(s) = \langle g(s)\zeta D, \omega' \rangle(s) \quad (D \in \mathfrak{L}(G), \omega' \in \mathfrak{I}(G'), s \in G).$$

In particular, a rational homomorphism $G \to G'$ induces vector space homomorphisms $\mathfrak{L}(G) \to \mathfrak{L}(G')$ and $\mathfrak{I}(G') \to \mathfrak{I}(G)$ each of which is the transpose of the other.

If $\nu: G \to G'$ is a rational homomorphism, and if $\zeta': G' \to G''$ is a rational crossed homomorphism (G' operating on the connected algebraic group G'' by g'), then G operates on G'' by $g'\nu$, and $\zeta'\nu$ is a rational crossed homomorphism of G into G''; furthermore, $(\zeta'\nu)D = \zeta'(\nu D)$ $(D \in \mathfrak{L}(G))$.

We now apply these considerations to the differentials $\omega(\alpha) = \omega_{K/k}(\alpha)$ of K over k induced by an invariant differential ω on a connected algebraic group G; as before, G is defined over k, ω is defined over K, and $\alpha \in G_K$ (the group of points of G which are rational over K).

As examples of rational homomorphisms we have the two projections

$$\pi_j: G \times G \to G, \qquad \pi_j(s_1, s_2) = s_j,$$

the three injections

$$\iota_1 : G \to G \times G, \quad \iota_1 s = (s, 1),$$
$$\iota_2 : G \to G \times G, \quad \iota_2 s = (1, s),$$
$$\Delta : G \to G \times G, \quad \Delta s = (s, s),$$

the identity mapping $\iota : G \to G$, and the trivial endomorphism

$$\epsilon : G \to G, \qquad \epsilon s = 1.$$

Obviously $\pi_1\iota_1 = \pi_2\iota_2 = \pi_1\Delta = \pi_2\Delta = \iota$, and $\pi_1\iota_2 = \pi_2\iota_1 = \epsilon$; also $\epsilon D = 0$ $(D \in \mathfrak{L}(G))$. Furthermore, $\pi_1{}^*\mathfrak{U}(G)$ is the subfield of $\mathfrak{U}(G \times G)$ consisting of the rational functions f on $G \times G$ such that $f(s_1, s_2)$ is independent of s_2, and similarly for $\pi_2{}^*\mathfrak{U}(G)$, so that $\mathfrak{U}(G \times G)$ is precisely the compositum

$$\mathfrak{U}(G \times G) = \pi_1{}^*\mathfrak{U}(G) \cdot \pi_2{}^*\mathfrak{U}(G). \tag{5}$$

For any $D \in \mathfrak{L}(G)$,

$$(\Delta D - \iota_1 D - \iota_2 D)_1 \pi_1{}^* f = (\pi_1(\Delta D - \iota_1 D - \iota_2 D)_1) f$$
$$= (\pi_1 \Delta D - \pi_1\iota_1 D - \pi_1\iota_2 D)_1 f = (\iota D - \iota D - \epsilon D)_1 f = 0$$

for all $f \in \mathfrak{o}_1$, so that

$$((\Delta D - \iota_1 D - \iota_2 D)\pi_1{}^* f)(s, t) = (\Delta D - \iota_1 D - \iota_2 D)_{(s,t)} \pi_1{}^* f = 0$$

provided $f \in \mathfrak{o}_s$; choosing (s, t) generic on $G \times G$ over a common field of definition of G, of D, and of f, we see that then $f \in \mathfrak{o}_s$ and

$$((\Delta D - \iota_1 D - \iota_2 D)\pi_1{}^* f)(s, t) = 0;$$

thus $(\Delta D - \iota_1 D - \iota_2 D)\pi_1{}^* f = 0$ for every $f \in \mathfrak{U}(G)$. Similarly

$$(\Delta D - \iota_1 D - \iota_2 D)\pi_2{}^* f = 0$$

for every $f \in \mathfrak{U}(G)$, so that by (5) $\Delta D = \iota_1 D + \iota_2 D$ $(D \in \mathfrak{L}(G))$.

Now, for each $t \in G$ let $\tau(t)$ denote the inner automorphism of G defined by t: $\tau(t)s = tst^{-1}$. Then G operates on itself by τ, and the reciprocation mapping

$$\rho : G \to G, \qquad \rho(s) = s^{-1}$$

is a rational crossed homomorphism. Also, $G \times G$ operates on G by $\tau\pi_2$, and the mapping

$$\chi : G \times G \to G, \qquad \chi(s, t) = st^{-1}$$

is a rational crossed homomorphism. Obviously $\chi\Delta = \epsilon$, $\chi\iota_1 = \iota$, and $\chi\iota_2 = \rho$. Therefore $0 = \epsilon D = \chi\Delta D = \chi\iota_1 D + \chi\iota_2 D = \iota D + \rho D$, so that

$$\rho D = -D \qquad (D \in \mathfrak{L}(G)).$$

It is easy to see that the two mappings $D \to \iota_j D$ of $\mathfrak{L}(G)$ into $\mathfrak{L}(G \times G)$ are injective. Also, if $D, E \in \mathfrak{L}(G)$ and $\iota_1 D + \iota_2 E = 0$ then $D = \iota D + \epsilon E = \pi_1 \iota_1 D + \pi_1 \iota_2 E = \pi_1(\iota_1 D + \iota_2 E) = 0$ and similarly $E = 0$. Therefore $\iota_1 \mathfrak{L}(G) \cap \iota_2 \mathfrak{L}(G) = \{0\}$, so that

$$\begin{aligned} \dim(\iota_1 \mathfrak{L}(G) + \iota_2 \mathfrak{L}(G)) &= \dim \iota_1 \mathfrak{L}(G) + \dim \iota_2 \mathfrak{L}(G) \\ &= 2 \dim \mathfrak{L}(G) = \dim \mathfrak{L}(G \times G), \end{aligned}$$

whence

$$\mathfrak{L}(G \times G) = \iota_1 \mathfrak{L}(G) + \iota_2 \mathfrak{L}(G) \quad \text{(direct sum)}.$$

Since $\chi \iota_1 D - \pi_1 \iota_1 D + \pi_2 \iota_1 D = \iota D - \iota D + \epsilon D = 0$ and $\chi \iota_2 D - \pi_1 \iota_2 D + \pi_2 \iota_2 D = \rho D - \epsilon D + \iota D = 0$, we conclude that

$$\chi D = \pi_1 \hat{D} - \pi_2 \hat{D} \qquad (\hat{D} \in \mathfrak{L}(G \times G)). \tag{6}$$

For any $\alpha, \beta \in G_K$ (α, β) is a point of $(G \times G)_K$ so that for any $\mathfrak{d} \in \mathfrak{D}(K/k)$ we have the tangent vector $\mathfrak{d}_{(\alpha,\beta)}$ to $G \times G$ at (α, β) and $\mathfrak{d}_{(\alpha,\beta)}$ is rational over K; also, there is a unique $D \in \mathfrak{L}(G \times G)$ with $\hat{D}_{(\alpha,\beta)} = \mathfrak{d}_{(\alpha,\beta)}$. Since χ is obviously defined over k it follows that $\langle \mathfrak{d}, \omega(\alpha\beta^{-1}) \rangle = \langle \mathfrak{d}, \omega(\chi(\alpha, \beta)) \rangle = \langle \mathfrak{d}, (\chi^* \omega)(\alpha, \beta) \rangle = \langle \mathfrak{d}_{(\alpha,\beta)}, \chi^* \omega_{(\alpha,\beta)} \rangle = \langle D, \chi^* \omega \rangle (\alpha, \beta)$; by the general formula (4) this equals $\langle \tau(\beta) \chi \hat{D}, \omega \rangle (\alpha, \beta)$, and by (6) this in turn equals

$$\begin{aligned} \langle \tau(\beta)(\pi_1 \hat{D} - \pi_2 \hat{D}), \omega \rangle (\alpha, \beta) &= \langle D, \pi_1{}^* \tau(\beta)^* \omega - \pi_2{}^* \tau(\beta)^* \omega \rangle (\alpha, \beta) \\ &= \langle \mathfrak{d}_{(\alpha,\beta)}, (\pi_1{}^* \tau(\beta)^* \omega)_{(\alpha,\beta)} - (\pi_2{}^* \tau(\beta)^* \omega)_{(\alpha,\beta)} \rangle \\ &= \langle \mathfrak{d}, (\pi_1{}^* \tau(\beta)^* \omega)(\alpha, \beta) - \pi_2{}^* \tau(\beta)^* \omega)(\alpha, \beta) \rangle \\ &= \langle \mathfrak{d}, (\tau(\beta)^* \omega)(\alpha) - (\tau(\beta)^* \omega)(\beta). \end{aligned}$$

Thus, $\omega(\alpha\beta^{-1}) = (\tau(\beta)^* \omega)(\alpha) - (\tau(\beta)^* \omega)(\beta)$. Applying this formula to $\tau(\beta^{-1})^* \omega$ instead of ω, and then replacing α by $\alpha\beta$, we obtain the desired formula (3) of the introduction.

2. *G*-extensions and *G*-primitives. In this section $\mathcal{F}$ denotes a commutative differential field of characteristic 0 with algebraically closed field of constants $\mathcal{C}$; the derivation operators of $\mathcal{F}$ are denoted by $\delta_1, \cdots, \delta_m$. $\mathcal{U}$ denotes a fixed universal extension of $\mathcal{F}$ (in the sense of [2] p. 768) and $\mathcal{K}$ denotes the field of constants of $\mathcal{U}$. We identify the derivation operators $\delta_1, \cdots, \delta_m$ in an obvious way with derivations of $\mathcal{U}$ over $\mathcal{K}$.

$\mathcal{U}$ is an algebraically closed extension of $\mathcal{F}$ of infinite transcendence degree, and therefore can be used as a universal domain for algebraic geometry. Whenever we introduce algebraico-geometric notions it always is in the algebraic geometry based on $\mathcal{U}$ as universal domain. The algebraic varieties we

have occasion to introduce, other than points, are actually defined over $\mathfrak{k}$; if ω is a differential on such a variety, the symbol $\omega(\alpha)$ always denotes the differential $\omega_{\mathcal{U}/\mathcal{K}}(\alpha)$ of $\mathcal{U}$ over $\mathcal{K}$ induced by ω at α, so that α can be any simple point of the variety at which ω is finite (and therefore, in the case of an invariant differential on a group variety, can be any point whatsoever).

Let G be a connected algebraic group defined over $\mathfrak{k}$; denote by $G_{\mathcal{K}}$ the group consisting of the points of G which are rational over $\mathcal{K}$.

It is apparent from the definition of induced differentials in § 1 that *if* $\gamma \in G$, *then a necessary and sufficient condition that* $\gamma \in G_{\mathcal{K}}$ *is that* $\langle \delta_i, \omega(\gamma) \rangle = 0$ *for every index* i *and every invariant differential* ω *on* G.

It follows from this that *if* $\alpha, \beta \in G$ *then a necessary and sufficient condition that* $\beta\alpha^{-1} \in G_{\mathcal{K}}$ *is that* $\langle \delta_i, \omega(\alpha) \rangle = \langle \delta_i, \omega(\beta) \rangle$ *for every index* i *and every invariant differential* ω *on* G. Indeed, setting $\gamma = \beta\alpha^{-1}$ we have, by formula (3),

$$\begin{aligned} \langle \delta_i, \omega(\beta) \rangle &= \langle \delta_i, \omega(\gamma\alpha) \rangle \\ &= \langle \delta_i, (\tau(\alpha^{-1})^*\omega)(\gamma) \rangle + \langle \delta_i, \omega(\alpha) \rangle, \end{aligned}$$

and as ω runs through the set of invariant differentials on G so does $\tau(\alpha^{-1})^*\omega$.

It is also obvious from the definition of induced differentials that *if* ω *is an invariant differential on* G *which is defined over* $\mathfrak{k}$ *and* $\alpha \in G$, *then* $\langle \delta_i, \omega(\alpha) \rangle \in \mathfrak{k}\langle\alpha\rangle$ $(1 \leqq i \leqq m)$. *If in addition* σ *is an isomorphism of the differential field* $\mathfrak{k}\langle\alpha\rangle$ *over* $\mathfrak{k}$ *onto an extension of* $\mathfrak{k}$ *in* $\mathcal{U}$, *then* $\sigma\langle \delta_i, \omega(\alpha) \rangle = \langle \delta_i, \omega(\sigma\alpha) \rangle$ $(1 \leqq i \leqq m)$. To prove the second point we observe that there exist uniformizing coordinates $x_1, \cdots, x_n$ on G at α with each $x_j \in \mathfrak{k}(G)$, and we may write $\omega = \sum f_j\, dx_j$ with each $f_j \in \mathfrak{o}_\alpha \cap \mathfrak{k}(G)$; then

$$\langle \delta_i, \omega(\alpha) \rangle = \sum_{1 \leqq j \leqq n} f_j(\alpha)\, \delta_i x_j(\alpha),$$

and the result follows.

Let α be a G-primitive over $\mathfrak{F}$. We recall from the introduction that this means: 1° the field of constants of $\mathfrak{F}\langle\alpha\rangle$ is $\mathfrak{k}$; 2° $\langle \delta_i, \omega(\alpha) \rangle \in \mathfrak{F}$ for each index i and every invariant differential ω on G defined over $\mathfrak{k}$. Then, for each isomorphism σ of $\mathfrak{F}\langle\alpha\rangle$ over $\mathfrak{F}$ onto an extension of $\mathfrak{F}$ in $\mathcal{U}$, $\langle \delta_i, \omega(\sigma\alpha) \rangle = \sigma\langle \delta_i, \omega(\alpha) \rangle = \langle \delta_i, \omega(\alpha) \rangle$, so that the point $\gamma(\sigma) = \alpha^{-1} \cdot \sigma\alpha$ is in $G_{\mathcal{K}}$. Therefore $\mathfrak{F}\langle\alpha\rangle \cdot \sigma(\mathfrak{F}\langle\alpha\rangle) = \mathfrak{F}\langle\alpha, \sigma\alpha\rangle = \mathfrak{F}\langle\alpha, \gamma(\sigma)\rangle = \mathfrak{F}\langle\sigma\alpha, \gamma(\sigma)\rangle$, so that $\mathfrak{F}\langle\alpha\rangle$ is strongly normal over $\mathfrak{F}$, and ([2] p. 768, cor. 5) the field of constants of $\mathfrak{F}\langle\alpha\rangle \cdot \sigma(\mathfrak{F}\langle\alpha\rangle)$ is $\mathfrak{k}(\gamma(\sigma))$. Identifying the isomorphisms of $\mathfrak{F}\langle\alpha\rangle$ over $\mathfrak{F}$ with the unique automorphisms of $\mathfrak{F}\langle\alpha\rangle \cdot \mathcal{K}$ over $\mathfrak{F} \cdot \mathcal{K}$ to which they extend ([2], Ch. II, § 3), we find that $\alpha \cdot \gamma(\sigma\tau) = \sigma\tau\alpha = \sigma(\alpha \cdot \gamma(\tau)) = \alpha \cdot \gamma(\sigma) \cdot \gamma(\tau)$, so that $\gamma(\sigma\tau) = \gamma(\sigma)\gamma(\tau)$. Since $\gamma(\sigma) = 1$ only if $\sigma = 1$,

the mapping $\sigma \to \gamma(\sigma)$ is a birational isomorphism of the Galois group of $\mathfrak{F}\langle\alpha\rangle$ over $\mathfrak{F}$ onto an algebraic subgroup of $G_{\mathfrak{K}}$. In particular, *every G-primitive extension of $\mathfrak{F}$ is a G-extension of $\mathfrak{F}$.*

Conversely, suppose that we are given a G-extension of $\mathfrak{F}$, that is, a strongly normal extension $\mathfrak{G}$ of $\mathfrak{F}$ such that there exists an injective rational homomorphism $\gamma: \mathfrak{G} \to G_{\mathfrak{K}}$, $\mathfrak{G}$ denoting the Galois group of $\mathfrak{G}$ over $\mathfrak{F}$.

If γ splits over $\mathfrak{G}$, i.e. if there exists a point $\alpha \in G_{\mathfrak{C}}$ such that $\gamma(\sigma) = \alpha \cdot \sigma\alpha^{-1}$ $(\sigma \in \mathfrak{G})$, then

$$\sigma\langle\delta_i, \omega(\alpha)\rangle = \langle\delta_i, \omega(\sigma\alpha)\rangle = \langle\delta_i, \omega(\gamma(\sigma^{-1})\alpha)\rangle$$
$$= \langle\delta_i, (\tau(\alpha^{-1})^*\omega)(\gamma(\sigma^{-1}))\rangle + \langle\delta_i, \omega(\alpha)\rangle = \langle\delta_i, \omega(\alpha)\rangle \quad (\sigma \in \mathfrak{G}),$$

so that $\langle\delta_i, \omega(\alpha)\rangle \in \mathfrak{F}$ for each index i and each invariant differential ω on G defined over $\mathfrak{C}$, i.e. α is a G-primitive over $\mathfrak{F}$. Since γ is injective, $\sigma\alpha = \alpha$ only if $\sigma = 1$, so that $\mathfrak{F}\langle\alpha\rangle = \mathfrak{G}$. Thus we have proved, in particular, that *if $\mathfrak{G}$ is a G-extension of $\mathfrak{F}$ with Galois group $\mathfrak{G}$ and if $H^1(\mathfrak{G}, G) = 1$ then $\mathfrak{G}$ is a G-primitive extension of $\mathfrak{F}$.*

If we do not assume that γ splits over $\mathfrak{G}$, but suppose instead that G is commutative, then ([3] p. 887), for some integer $h > 0$, γ^h splits over $\mathfrak{G}$: there exist a point $\alpha \in G_{\mathfrak{C}}$ such that $\gamma(\sigma)^h = \alpha \cdot \sigma\alpha^{-1}$ $(\sigma \in \mathfrak{G})$. The same reasoning as above then proves: *If G is commutative and $\mathfrak{G}$ is a G-extension of $\mathfrak{F}$ with Galois group $\mathfrak{G}$, then there exist a G-primitive extension $\mathfrak{H}$ of $\mathfrak{F}$ and an integer $h > 0$ such that $\mathfrak{F} \subset \mathfrak{H} \subset \mathfrak{G}$ and the Galois group of $\mathfrak{G}$ over $\mathfrak{H}$ is the finite group $\mathfrak{G}(h)$ of all elements σ of $\mathfrak{G}$ with $\sigma^h = 1$.*

3. Abelian extensions and abelian functions. In this section we consider the classical case in which $\mathfrak{F}$ is a differential field consisting of functions meromorphic in some region of complex m-space $\boldsymbol{C}^m$, the derivation operators being the partial derivations $\partial/\partial x_1, \cdots, \partial/\partial x_m$ with respect to the m coordinate functions $x_1, \cdots, x_m$, and the field of constants of $\mathfrak{F}$ being the complex number field $\boldsymbol{C}$. We seek to characterize the A-primitive extensions of $\mathfrak{F}$ in function-theoretic terms, A denoting an arbitrary abelian variety defined over $\boldsymbol{C}$.

To this end we recall certain facts about abelian varieties defined over $\boldsymbol{C}$ and abelian functions. If Γ is a free abelian subgroup of $\boldsymbol{C}^n$ with $2n$ generators, the field $\mathcal{A}$ of all meromorphic functions on $\boldsymbol{C}^n$ admitting as periods the elements of Γ is called the abelian function field with period group Γ. $\mathcal{A}$ is said to be degenerate if by means of some invertible linear transformation on $\boldsymbol{C}^n$ the elements of $\mathcal{A}$ can all be expressed as meromorphic functions of fewer than n variables, that is, if there exist complex numbers

$k_1, \cdots, k_n$ not all 0 such that $\sum k_j \partial f/\partial z_j = 0$ $(f \in \mathcal{A})$, $z = (z_1, \cdots, z_n)$ denoting the usual coordinate functions on $\boldsymbol{C}^n$, and $\mathcal{A}$ is said to be nondegenerate otherwise. There are well known necessary and sufficient conditions on Γ for $\mathcal{A}$ to be nondegenerate (the period relations). When $\mathcal{A}$ is nondegenerate then Γ is of rank $2n$ over the field of real numbers, and there exist an abelian variety A of dimension n defined over $\boldsymbol{C}$ and a biholomorphic group isomorphism $\boldsymbol{C}^n/\Gamma \approx A_{\boldsymbol{C}}$ (these groups being endowed with their usual complex analytic structures). The composite of the canonical projection $\boldsymbol{C}^n \to \boldsymbol{C}^n/\Gamma$ and this isomorphism is a surjective holomorphic homomorphism $p: \boldsymbol{C}^n \to A_{\boldsymbol{C}}$ with kernel Γ such that $\mathcal{A} = \boldsymbol{C}(p(z))$. Conversely, if A is any abelian variety of dimension n defined over $\boldsymbol{C}$, there exist a nondegenerate abelian function field $\mathcal{A}$ and a surjective holomorphic homomorphism $p: \boldsymbol{C}^n \to A_{\boldsymbol{C}}$ exactly as above. These facts, or a reasonable facsimile of them, can be found in Chapter VI of Weil's book [6] (see especially his Theorem 5, page 130).

Now, $\mathcal{A}$ is obviously a differential field relative to the derivation operators $\partial/\partial z_1, \cdots, \partial/\partial z_n$, the field of constants of $\mathcal{A}$ being $\boldsymbol{C}$, and for each $k \in \boldsymbol{C}^n$ the translation mapping $f(z) \to f(z+k)$ $(f(z) \in \mathcal{A})$ is an automorphism of this differential field over $\boldsymbol{C}$. It follows by § 2 that if $(\omega_1, \cdots, \omega_n)$ denotes a basis of the space of invariant differentials on A defined over $\boldsymbol{C}$ then this automorphism maps $\langle \partial/\partial z_j, \omega_{j'}(p(z)) \rangle$ onto $\langle \partial/\partial z_j, \omega_{j'}(p(z+k)) \rangle$, which by § 2 equals $\langle \partial/\partial z_j, \omega_{j'}(p(z)) \rangle + \langle \partial/\partial z_j, \omega_{j'}(p(k)) \rangle = \langle \partial/\partial z_j, \omega_{j'}(p(z)) \rangle$. Thus $\langle \partial/\partial z_j, \omega_{j'}(p(z)) \rangle$ is invariant under every translation, that is, is a complex constant, which we denote by $a_{jj'}$ $(1 \leqq j \leqq n, 1 \leqq j' \leqq n)$. In other words, $p(z)$ *is an* A*-primitive over* $\boldsymbol{C}$. Because $\mathcal{A}$ is nondegenerate, the matrix $(a_{jj'})$ is invertible.

Shifting our attention to the differential field $\mathcal{F}$ of functions of m complex variables $x_1, \cdots, x_m$ meromorphic in some region D of $\boldsymbol{C}^m$ (m possibly different from n), let $\zeta(x) = (\zeta_1(x_1, \cdots, x_m), \cdots, \zeta_n(x_1, \cdots, x_m))$ be a sequence of n functions of $x = (x_1, \cdots, x_m)$ meromorphic in some subregion of D. Then

$$\langle \partial/\partial x_i, \omega_{j'}(p(\zeta(x))) \rangle = \sum_j \partial \zeta_j(x)/\partial x_i \cdot \langle \partial/\partial z_j, \omega_{j'}(p(z)) \rangle = \sum_j \partial \zeta_j(x)/\partial x_i \cdot a_{jj'},$$

so that we may write the matrix equation

$$(*) \qquad (\langle \partial/\partial x_i, \omega_{j'}(p(\zeta(x))) \rangle) = (\partial \zeta_j(x)/\partial x_i)\,(a_{jj'}).$$

Therefore *if each* $\zeta_j(x)$ *is primitive over* $\mathcal{F}$, *i.e.* $\partial \zeta_j(x)/\partial x_i \in \mathcal{F}$ $(1 \leqq i \leqq m, 1 \leqq j \leqq n)$, *then* $p(\zeta(x))$ *is an* A*-primitive over* $\mathcal{F}$.

Conversely, let q be any A-primitive over $\mathcal{F}$, so that we may write

$$\langle \partial/\partial x_i, \omega_{j'}(q)\rangle = f_{ij'}(x) \in \mathfrak{F} \qquad (1 \leqq i \leqq m, 1 \leqq j \leqq n).$$

It is easy to see that for each index j' the m functions $f_{ij'}(x)$ satisfy the integrability conditions $\partial f_{ij'}(x)/\partial x_{i'} = \partial f_{i'j'}(x)/\partial x_i$ $(1 \leqq i \leqq m, 1 \leqq l' \leqq m)$, and therefore for each j the m functions $g_{ij}(x)$ defined by the matrix equation $(g_{ij}(x)) = (f_{ij'}(x))(a_{jj'})^{-1}$ satisfy similar integrability conditions. Consequently there exist functions $\zeta_j(x)$ meromorphic in some subregion of D such that $\partial\zeta_j(x)/\partial x_i = g_{ij}(x)$ $(1 \leqq i \leqq m, 1 \leqq j \leqq n)$. Therefore by (7)

$$(\langle \partial/\partial x_i, \omega_{j'}(p(\zeta(x)))\rangle) = (g_{ij}(x))(a_{jj'}) = (f_{ij'}(x)) = (\langle \partial/\partial x_i, \omega_{j'}(q)\rangle).$$

It follows by § 2 that $q \cdot p(\zeta(x))^{-1} \in A_{\boldsymbol{C}}$, i.e. $q \cdot p(\zeta(x))^{-1} = p(k)$ for some $k = (k_1, \cdots, k_n) \in \boldsymbol{C}^n$, whence $q = p(\zeta(x) + k)$. Since each $\zeta_j(x) + k_j$ is primitive over $\mathfrak{F}$, we have the following result: *In the classical case of a differential field $\mathfrak{F}$ of functions meromorphic in a region of complex m-space $\boldsymbol{C}^m$ with field of constants $\boldsymbol{C}$, a necessary and sufficient condition that a point $q \in A$ be an A-primitive over $\mathfrak{F}$ is that $q = p(\zeta)$, where $\zeta = (\zeta_1, \cdots, \zeta_n)$ and each ζ_j is primitive over $\mathfrak{F}$.* This means that the A-primitive extensions of $\mathfrak{F}$ are precisely the extensions of the form $\mathfrak{F}\langle p(\zeta)\rangle = \mathfrak{F}(p(\zeta))$, and therefore: *Every A-primitive extension of $\mathfrak{F}$ is obtained by adjoining to $\mathfrak{F}$ finitely many functions $f_1(\zeta), \cdots, f_r(\zeta)$, where $f_1(z), \cdots, f_r(z)$ are abelian functions which generate the nondegenerate abelian function field $\mathfrak{A}$ and $\zeta = (\zeta_1, \cdots, \zeta_n)$ is a sequence of n functions which are primitive over $\mathfrak{F}$.*

COLUMBIA UNIVERSITY.

REFERENCES.

[1] Iacopo Barsotti, "Un teorema di struttura per le varieta gruppali," *Rendiconti dell'Academia Nazionale dei Lincei*, vol. 18 (1955), pp. 43-50.

[2] E. R. Kolchin, "Galois theory of differential fields," *American Journal of Mathematics*, vol. 75 (1953), pp. 753-824.

[3] ———, "On the Galois theory of differential fields," *ibid.*, vol. 77 (1955), pp. 868-894.

[4] Maxwell Rosenlicht, "Some basic theorems on algebraic groups," *ibid.*, vol. 78 (1956), pp. 401-443.

[5] ———, "A note on derivations and differentials on algebraic varieties," *Portugaliae mathematica*, vol. 16 (1957), pp. 43-55.

[6] André Weil, *Introduction à l'étude des variétés kaehlériennes* (Actualités scientifiques et industrielles 1267), 1958.

THE ALGEBRAIC DETERMINATION OF THE GENUS OF KNOTS.*

By LEE NEUWIRTH.[1]

1. Introduction. The genus of a tame knot in S^3 was first defined by Seifert [1]. Although it was defined purely geometrically, he showed that the degree of the Alexander Polynomial, a purely algebraically defined invariant, is at most twice the genus. This inequality relating a geometric invariant to an algebraic invariant was sharpened by Crowell [2] and Murasugi [3] to an equality for alternating knots, which are defined geometrically.

In the course of investigating the commutator subgroup of knot groups I was able to prove a structure theorem which shows that *the genus of a knot is, under a certain simple algebraic condition, determined algebraically by the knot group.* Furthermore, this algebraic condition is often easily checked, and if satisfied, the genus is effectively computable.

Using a result of E. Rapaport [4] and the structure theorem mentioned above, equality between the degree of the Alexander Polynomial and twice the genus is established under an algebraic hypothesis quite different from the geometric hypothesis of Crowell and Murasugi.

A number of corollaries to the structure theorem will be proved, some of which relate to the center of a knot group, and constitute a start in the direction of proving the following:

Conjecture 1. A knot group G is the group of a torus knot if and only if G has a non-trivial center.

2. Hypothesis and notation. Throughout this paper k will denote an arbitrary but fixed non-trivial, polygonal knot in S^3.

By the group of the knot k, is meant $\pi_1(S^3 - k)$, which will be denoted G.

g will denote the genus of k.

Unless the base point is critical $\pi_1(X, x_0)$ will be denoted $\pi_1(X)$.

* Received November 11, 1959.

[1] The helpful suggestions of the referee are gratefully acknowledged. Part of this paper was prepared while the author held N. S. F. Post-doctoral Fellowship 49164.

The free product with amalgamation of two groups E, F on subgroups B, A respectively, will be denoted $E *_C F$, where $C \approx B \approx A$.

$\sim$ will denote "is homotopic to."

3. A construction. The infinite cyclic unbranched covering of $S^3 - k$ will be constructed in this section.

Let $S \subset S^3$ denote a polyhedral orientable surface of minimal genus (g) spanned by k [1].

Let X denote the unbranched covering of $S^3 - k$ corresponding to $[G, G]$.

Let U be an open regular neighborhood (in the sense of [5]) of the interior of S. Then $\dot{U} \supset k$, and $\bar{U}$ is a manifold with boundary [5]. Since S is orientable $\dot{U} - k$ has two polyhedral components, call them S_1 and S_2. Notice that genus of $\bar{S}_1 =$ genus of $\bar{S}_2 =$ genus of $S = g$.

X will be constructed by matching copies of $S^3 - (U \cup k)$ along S_1 and S_2. More precisely, let $X_0 = S^3 - (U \cup k)$. Let $\{X_i\}^{+\infty}_{i=-\infty}$ denote a countable collection of disjoint copies of X_0, where X_i may be identified with X_0 by a natural homeomorphism $f_i : X_i \to X_0$. In each X_i let ${}_iS_1$, ${}_iS_2$ denote $f_i^{-1}(S_1)$, $f_i^{-1}(S_2)$ respectively. Since U is a regular neighborhood of the interior of S, the interior of S is a deformation retract of $\bar{U} - k$ (see [5], p. 291); furthermore, this deformation retraction may be adjusted so as to map each S_j homeomorphically onto the interior of S; let g_j be this homeomorphism for $j = 1$ or 2.

Now match each X_i with X_{i-1} along ${}_{i-1}S_1$, ${}_iS_2$ as follows. Consider the space X' obtained from $\{X_i\}^{+\infty}_{i=-\infty}$ by factoring by the relation,

$$x = f^{-1}_{i+1} g_2^{-1} g_1 f_i(x) \quad \text{if } x \in {}_iS_1 \text{ for some } i.$$

This matches ${}_iS_1$ with ${}_{i+1}S_2$ for all i. X' may be pictured with the help of this diagram.

$$\begin{array}{ccc|ccc|ccc} & X_{i-1} & & & X_i & & & X_{i+1} & \\ & & & & & & & & \\ & {}_{i-1}S_1 & & {}_iS_2 & & {}_iS_1 & & {}_{i+1}S_2 & \end{array}$$

It must be shown that X' is homeomorphic to X. Recall $G/[G, G] \approx Z$. Clearly one sheet of X is homeomorphic to $S^3 - (U \cup k \cup S_1)$. The elements, t^n, of the group of covering translations of X thus correspond in a one to one manner to the maps $(f^{-1}_{i+1} f_i)^n$. Since X' covers $S^3 - k$ in the obvious way, and since X' has an infinite cyclic group of covering translations $(\{f^{-1}_{i+1} f_i\}_{i \in Z})$ it follows that X' is homeomorphic to X. It will be assumed from now on that the X_i are imbedded in X'.

Two lemmas are needed before we can proceed to calculate $\pi_1(X) = [G, G]$.

4. Two lemmas.

LEMMA 1. *The inclusion map* $h: S_1 \to \dot{U}$ *induces a monomorphism* $h^*: \pi_1(S_1) \to \pi_1(\dot{U})$.

Proof. Since k is knotted, S_i is of genus greater than zero, hence no power of k, considered as an element of $\pi_1(S_i)$ is null homotopic. $\dot{U}$ is constructed by identifying $\bar{S}_1$ with $\bar{S}_2$ along k. A simple application of the Van Kampen Theorem thus gives

$$\pi_1(U) \approx \pi_1(\bar{S}_1) \underset{\pi_1(k)}{*} \pi_1(\bar{S}_2).$$

Since the inclusion $S_1 \subset \bar{S}_1$ clearly induces an isomorphism of the fundamental group, $\pi_1(S_1)$ is imbedded monomorphically in $\pi_1(\dot{U})$ by a map induced by the inclusion h.

LEMMA 2. *The inclusion map* $i: S_1 \to (S^3 - U)$ *induces a monomorphism* $i^*: \pi_1(S_1) \to \pi_1(S^3 - U)$.

Proof. Suppose the lemma is false, and that α is a closed curve on S_1, such that $\alpha \sim 0$ in $S^3 - U$, $\alpha \not\sim 0$ on S_1. According to Lemma 1, $\alpha \not\sim 0$ on $\dot{U}$. Since S_1 is polyhedral, α may be assumed polygonal so that $(S^3 - U)$ and α satisfy the hypothesis of the Loop Theorem ([6], Theorem 15.1 and Theorem 1), thus we may assume that α is simple. According to Dehn's Lemma [7], α bounds a non-singular polyhedral disc in $S^3 - U$. If we make a cut (Umschaltung) along this disc and its boundary we obtain a new surface S'_1 which is bounded by k. If the curve α separates S_1, then because $\alpha \not\sim 0$ on S_1, the new surface S'_1 has lower genus than $\bar{S}_1$, which contradicts the assumption that the genus of $\bar{S}_1$ is minimal. If α does not separate S_1, then compare the Euler characteristic of $\bar{S}_1$, $\chi(\bar{S}_1)$, with that of $\bar{S}'_1$. Since the cut adds one vertex, one edge, and two faces, $\chi(S'_1) - \chi(\bar{S}_1) = 2$, hence S'_1 has lower genus than $\bar{S}_1$, so we again arrive at a contradiction. The existence of the curve α thus leads to a contradiction, so the lemma is proved.

Remark. Lemmas 1 and 2 obviously remain valid if S_2 is substituted for S_1. Lemma 2 also remains valid if $S^3 - (U - k)$ is substituted for $S^3 - U$.

5. The structure theorem.

THEOREM 1. *If* $[G, G]$ *is finitely generated, it is free of rank* $2g$, *where* g *is the genus of* k.

If $[G, G]$ is not finitely generated, then either it is:

A) *a non-trivial free product with amalgamation on a free rank $2g$, and may be written in the form*

$$\cdots * \underset{F_{2g}}{A} * \underset{F_{2g}}{A} * \underset{F_{2g}}{A} * \underset{F_{2g}}{A} * \underset{F_{2g}}{A} * \cdots,$$

where F_{2g} is free of rank $2g$, and the amalgamations are all proper and identical, or

B) *locally free, and a direct limit of free groups of rank $2g$.*

Proof. By virtue of Lemma 2, the last remark in 4 and the identification of ${}_iS_1$ and ${}_{i+1}S_2$, the following diagram is valid for every i,

$$\pi_1(X_i) \xleftarrow[{}_if_1]{} \pi_1({}_iS_1) \xrightarrow[{}_{i+1}f_2]{} \pi_1(X_{i+1})$$

where the ${}_if_j$ are monomorphisms.

By a simple application of the Van Kampen theorem, the fundamental group of $X_i \cup X_{i+1}$ is the direct limit of the above system. This direct limit is a free product with amalgamated subgroup, $\pi_1(X_i) \underset{\pi_1(S)}{*} \pi_1(X_{i+1})$.

Let
$$Y_n = X_0 \cup X_1 \cup \cdots X_{n-1} \cup X_n, \qquad n \geqq 0$$
$$Y_{-n} = X_{-1} \cup X_{-2} \cup \cdots X_{-n+1} \cup X_{-n}, \qquad n \geqq 1$$
$$Y_\infty = \bigcup_{i=0}^{\infty} X_i \qquad Y_{-\infty} = \bigcup_{i=-1}^{-\infty} X_i.$$

Using the fact that each factor in a free product with amalgamation is contained as a subgroup in the free product with amalgation ([8], p. 32) and proceeding inductively it is clear that from the above diagram one obtains,

$$(1) \qquad \pi_1(Y_n) = \pi_1(Y_{n-1}) \underset{\pi_1(S)}{*} \pi_1(X_n)$$
$$= (((\pi_1(X_0) \underset{\pi_1(S)}{*} \pi_1(X_1)) \underset{\pi_1(S)}{*} \pi_1(X_2)) \cdots \underset{\pi_1(S)}{*} \pi_1(X_{n-1})) \underset{\pi_1(S)}{*} \pi_1(X_n).$$

$$(2) \qquad \pi_1(Y_{-n}) = \pi_1(Y_{-n+1}) \underset{\pi_1(S)}{*} \pi_1(X_{-n})$$
$$= (((\pi_1(X_{-1}) \underset{\pi_1(S)}{*} \pi_1(X_{-2})) \underset{\pi_1(S)}{*} \pi_1(X_{-3})) \cdots \underset{\pi_1(S)}{*} \pi_1(X_{-n+1})) \underset{\pi_1(S)}{*} \pi_1(X_{-n}).$$

$\pi_1(Y_\infty) = \varinjlim_{n \geqq 0} (Y_n, \phi_n)$, where ϕ_n denotes the inclusion isomorphism of $\pi_1(Y_n)$ in $\pi_1(Y_{n+1})$ ([8], p. 32), and

$\pi_1(Y_{-\infty}) = \varinjlim_{n \geqq 0} (Y_{-n}, \rho_{-n})$, where ρ_{-n} denotes the inclusion isomorphism of $\pi_1(Y_{-n+1})$ in $\pi_1(Y_{-n})$ ([8], p. 32).

It follows then that $\pi_1(Y_0) \subset \pi_1(Y_\infty)$, and $\pi_1(Y_{-1}) \subset \pi_1(Y_{-\infty})$. This fact and the diagram above imply that

$$\pi_1(X) = \pi_1(Y_{-\infty} \cup Y_\infty) = \pi_1(Y_{-\infty}) \underset{\pi_1(S)}{*} \pi_1(Y_\infty).$$

Note that if k is of genus g, then $\pi_1(S)$ is free of rank $2g$.

Suppose one of the maps, ${}_0f_i\colon \pi_1({}_0S_1) \to \pi_1(X_0)$ is not onto, say for $i=1$. Then no ${}_jf_i$ will be onto, so that

$$\pi_1(Y_\infty) \subsetneqq \pi_1(Y_{-1} \cup Y_\infty) \subsetneqq \pi_1(Y_{-2} \cup Y_\infty) \cdots$$

and

$$\pi_1(X) = \bigcup_{n=1}^{\infty} \pi_1(Y_{-n} \cup Y_\infty)$$

so that $\pi_1(X)$ is not finitely generated. But if ${}_0f_i\colon \pi_1({}_0S_i) \to \pi_1(X_0)$ is onto for $i = 1, 2$, then all ${}_jf_i$ are onto so that

$$\pi_1(S) \approx \pi_1(X_0) \approx \pi_1(Y_1) \approx \pi_1(Y_n) \approx \pi_1(Y_\infty) \approx \pi_1(Y_{-1} \cup Y_\infty)$$
$$\approx \pi_1(Y_{-n} \cup Y_\infty) \approx \pi_1(Y_{-\infty} \cup Y_\infty) \approx \pi_1(X)$$

and $\pi_1(X)$ is free of rank $2g$. Hence if $[G, G] = \pi_1(X)$ is finitely generated ${}_0f_1$ and ${}_0f_2$ are onto and $\pi_1(X)$ is free of rank $2g$. This proves the first assertion of Theorem 1.

If neither of the mappings ${}_0f_i$ is onto, then $\pi_1(X) = \pi_1(Y_{-\infty}) \underset{\pi_1(S)}{*} \pi_1(Y_\infty)$ is a proper free product with amalgamation and may be written as $\varinjlim_{n>0} \pi_1(Y_n \cup Y_{-n})$, where each $\pi_1(Y_n \cup Y_{-n})$ is a proper free product with amalgamation of $\pi_1(Y_n)$ and $\pi_1(Y_{-n})$ on a free group of rank $2g$ isomorphic to $\pi_1(S)$, so by virtue of equations 1) and 2, A) is proven. Suppose one of the maps, ${}_0f_i$, is onto, say ${}_0f_2$, and the other, ${}_0f_1$, is not, then

$$\pi_1(Y_\infty \cup Y_{-n}) \subsetneqq \pi_1(Y_\infty \cup Y_{-n-1})$$

and

$$\pi_1(S) \approx \pi_1(Y_0) \approx \pi_1(Y_1) \approx \pi_1(Y_n) \approx \pi_1(Y_\infty).$$

Since Y_∞ is homeomorphic to $Y_{-n} \cup Y_\infty$,[2] $\pi_1(Y_\infty) \approx \pi_1(Y_{-n} \cup Y_\infty)$, and hence each of the groups in the direct system

$$\pi_1(Y_\infty) \subsetneq \pi_1(Y_\infty \cup Y_{-1}) \subsetneq \pi_1(Y_\infty \cup Y_{-2}) \cdots$$

is free of rank $2g$. Of course $\pi_1(X) = \varinjlim_{n>0} \pi_1(Y_\infty \cup Y_{-n})$. Since the mappings in this direct system are all inclusions, it follows that any finitely generated subgroup $H \subset \pi_1(X)$ lies in some $\pi_1(Y_\infty \cup Y_{-n})$ and so H is a subgroup of a free group and hence free [9]; in other words, $\pi_1(X) = [G, G]$ is locally free, and B) is proven. This completes the proof of the theorem.

6. Application. In [1] H. Seifert showed (Satz 4) that the torus knot of type (p, q) has genus $\frac{1}{2}(p-1)(q-1)$. Theorem 1 allows an alternative proof to Seifert's result, since the rank of the commutator subgroup of $G = (a, b : a^p = b^q)$ may be easily shown to be $(p-1)(q-1)$.

7. Corollaries. In this section a number of corollaries to Theorem 1 are given, many of which concern the center of a knot group.

COROLLARY 1. *If $[G, G]$ is finitely generated, then no orientable surface of minimal genus spanned by k is algebraically knotted.*[3]

Proof. If $[G, G]$ is finitely generated, $\pi_1(S) \approx \pi_1(X_0) \approx \pi_1(S^3 - S)$, hence $\pi_1(S^3 - S)$ is free.

COROLLARY 2. *The center of $[G, G]$ is trivial.*

Proof. If $[G, G]$ is finitely generated, it is free of rank > 1 according to Theorem 1, hence the corollary follows.

If $[G, G]$ is not finitely generated, it is either a free product with amalgamation on a centerless group and hence centerless according to [8, p. 32], or else locally free and non-abelian and hence centerless. Q. E. D.

COROLLARY 3. *If k is non-trivial, G contains a free group of rank n for any $n \leqq \infty$.*

Proof. $G \supset [G, G] \supset \pi_1(S)$ according to the proof of Theorem 1, and $\pi_1(S)$ is free of rank $2g$ ($g > 0$ since k is knotted). Since the free group of rank 2 contains a free group of countable rank, the corollary follows.

---

[2] The homeomorphism may be taken to be the covering translation t^n restricted to $Y_{-n} \cup Y_\infty$.

[3] A surface $S \subset S^3$ is algebraicallyknotted if $\pi_1(S^3 - S)$ is not free.

COROLLARY 4. *The comutator series of a non-cyclic knot group does not terminate after a finite number of steps.*

Proof. If the commutator series of a knot group terminated after a finite number of steps, the same would be true for any subgroup (see for example [8], p. 179). By Corollary 3 there exists a subgroup with a non-finite commutator series ([8], p. 36), hence the corollary is proved.

COROLLARY 5. *The center, Z, of a knot group, G, is cyclic.*

Proof. Corollary 2 implies $Z \cap [G, G] = 1$, hence

$$G/[G, G] \supset Z/Z \cap [G, G] = Z.$$

Since $G/[G, G]$ is cyclic, the corollary is proved.

Remark. Of course if Z is non-trivial, Z is infinite cyclic.

The existence of a non-trivial center in a knot group has a rather interesting consequence.

THEOREM 2. *If a knot group, G, has a non-trivial center, Z, then Z intersects every abelian subgroup of rank 2 non-trivially.*

Proof. According to Corollary 5, $Z = (z:)$. Suppose

$$G \supset H = (x:) \oplus (y:),$$

then the subgroup HZ is free abelian, and of rank 2 according to [10].

Let s and t generate HZ; then the following equations are valid for some integers m_i, n_i;

$$n_1 s + m_1 t = z,$$
$$n_2 s + m_2 t_2 = x,$$
$$n_3 s + m_3 t = y.$$

Since x and y generate H, and H is not cyclic, $n_3 x - n_2 y \neq 0$, $m_3 x - m_2 y \neq 0$; thus

$$(m_3 n_2 - m_2 n_3)(s) = m_3 x - m_2 y \neq 0,$$
$$(n_3 m_2 - n_2 m_3)(t) = n_3 x - n_2 y \neq 0.$$

Upon multiplying each of these equations by n_1 and m_1 respectively, we deduce

$$0 \neq K_1 z = K_2 x + K_3 y$$

so that $Z \cap H \neq 1$, which was to be proved.

Since $[G, G] \cap Z = 1$ by Corollary 2, Theorem 2 implies

COROLLARY 6. *If a knot group G has a non-trivial center, then $[G, G]$ has no abelian subgroups of rank 2.*

I have shown in [11] that *the group of any alternating knot which is not a torus knot has no center.* On the other hand it is easy to see that the group of any torus knot, $((a, b : a^m = b^n), (m, n) = 1)$ has an infinite cyclic center (generated by b^n). Thus Conjecture 1 is true for the class of alternating knots.

COROLLARY 7. *If $[G, G]$ is finitely generated, then the degree of the Alexander Polynomial is twice the genus of k.*

Proof. In [4] E. Rapaport has shown that when $[G, G]$ is free of rank $2g < \infty$, the degree of the Alexander polynomial is $2g$. This and Theorem 1 imply the corollary.

REFERENCES.

[1] H. Seifert, "Uber das Geschlect von Knoten," *Mathematische Annalen*, vol. 110 (1935), pp. 571-592.

[2] R. H. Crowell, "Genus of alternating link types," *Annals of Mathematics*, vol. 69 (1959), pp. 258-275.

[3] K. Murasugi, "On the genus of the alternating knot, I, II," *Journal of the Mathematical Society of Japan*, vol. 10 (1958), pp. 94-105, pp. 235-248.

[4] E. Rapaport, "On the commutator subgroup of a knot group," *Annals of Mathematics*, vol. 71 (1960), pp. 157-162.

[5] J. H. C. Whitehead, "Simplicial spaces, nuclei and m-groups," *Proceedings of the London Mathematical Society*, vol. 45 (1939), pp. 243-327.

[6] C. D. Papakyriakopoulos, "On solid tori," *ibid.*, (3), vol. 7 (1957), pp. 281-299.

[7] ———, "On Dehn's lemma and the asphericity of knots," *Annals of Mathematics*, vol. 66 (1957), pp. 1-26.

[8] K. A. Kurosh, *The Theory of Groups, Vol. II*, New York, 1955.

[9] O. Schreien, "Die Untergruppen der freien Gruppen," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 5 (1926), pp. 161-183.

[10] P. Conner, "On the action of a finite group on $S^n \times S^n$," *Annals of Mathematics*, vol. 66 (1957), pp. 586-588.

[11] L. Neuwirth, *Knot Groups*, Princeton University Doctoral Thesis, June, 1959.

AN ULTRAHYPERBOLIC EQUATION WITH AN INTEGRAL CONDITION.*[1]

By O. G. Owens.

Introduction. The ultrahyperbolic differential equation with four independent variables,

$$\partial^2 u/\partial t_1^2 + \partial^2 u/\partial t_2^2 = \partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2, \tag{0.1}$$

is of special importance in the investigation of the Hilbert problem of determining in 3-dimensional cartesian space all metrics for which the geodesics are straight lines. G. Hamel [1] showed its relationship to the Hilbert proposal and established for (0.1) a characteristic value problem, the data being analytic in some of the variables. F. John [2] has treated with elegance other characteristic value problems for the same equation, his data not being restricted by analyticity requirements. Furthermore, N. S. Piskunov [3] has treated the conical characteristic value problem for (0.1).

In 1923 J. Hadamard [4] asserted that the ultrahyperbolic equation does not seem to possess any "properly posed" limit problem; that is, an auxiliary limit condition which when taken in conjunction with the differential equation assures both the uniqueness and the existence of the solution. In 1946 I. G. Petrowsky [5] made a similar statement: "There is a sizable class of partial differential equations for which we do not know of any correctly posed limit problem. The so-called ultrahyperbolic equation seems to be one of these."

It seems to the author that the Hadamard-Petrowsky statement applies primarily to non-characteristic value problems and but one such problem, which seems to be a simple natural limit problem for (0.1), will be discussed in this paper. The problem will be concerned with solutions of (0.1) which are defined on the 4-dimensional cartesian region consisting of all points (x_1, x_2, t_1, t_2) where (t_1, t_2) is an arbitrary point of the cartesian (t_1, t_2)-space and (x_1, x_2) any point of a given bounded region $G(X)$ of (x_1, x_2)-space. These solutions are assumed to vanish for all (t_1, t_2) whenever (x_1, x_2) is at

* Received December 13, 1959.

[1] This work was supported by the National Science Foundation, Contract No. G6093.

the boundary $G^*(X)$ of $G(X)$. Moreover, the auxiliary limit condition required of the solution is that

$$(0.2) \qquad \int_0^\infty u\, ds = F(x_1, x_2, \theta) \qquad (t_1 + it_2 = se^{i\theta}),$$

where the prescribed function $F(x_1, x_2, \theta)$ is of period 2π in θ, regular on its domain, and vanishes with sufficient smoothness at the boundary $G^*(X)$. Subject to these restrictions it is shown that there exists a unique solution of (0,1), Theorem 3.1 and Theorem 12.1, and that the integral (0,2) is only conditional convergent, Theorem 12.2. An evident conclusion from these theorems and their proofs will be the existence of a bi-unique mapping of the above class of solutions of (0.1) onto denumerable sequences of harmonic functions defined on the unit circle.

1. Notation. $x = (x_1, x_2)$ and $t = (t_1, t_2)$ will denote any two points of the cartesian (x_1, x_2)-space and (t_1, t_2)-space, and $u = u(x, t) = u(x_1, x_2, t_1, t_2)$ any real-valued function of x and t. $G(X)$ will indicate any simply connected bounded region of (x_1, x_2)-space with boundary, $G^*(X)$, consisting of a simple closed curve formed from a finite number of analytic arcs. $G(T)$ signifies the entire (t_1, t_2)-space and $G = G(X) \times G(T)$ the 4-dimensional region consisting of all points $(x, t) = (x_1, x_2, t_1, t_2)$ with $x \in G(X)$ and $t \in G(T)$. The closure of any set E will be denoted by $\bar{E}$. The euclidean elements of area on $G(X)$ and $G(T)$ are abbreviated to $dx = dx_1 dx_2$ and $dt = dt_1 dt_2$, and the related double integrals are written as

$$\int_{G(X)} u\, dx = \int\int_{G(X)} u\, dx_1 dx_2, \qquad \int_{G(T)} u\, dt = \int\int_{G(T)} u\, dt_1 dt_2.$$

It is convenient to indicate the repeated Laplacian by

$$\Delta_x{}^l u = \Delta_x(\Delta_x{}^{l-1} u) \text{ or } \Delta_t{}^l u = \Delta_t(\Delta_t{}^{l-1} u) \qquad (l = 0, 1, \cdots),$$

where

$$\Delta_x u = \partial^2 u/\partial x_1{}^2 + \partial^2 u/\partial x_2{}^2, \qquad \Delta_t u = \partial^2 u/\partial t_1{}^2 + \partial^2 u/\partial t_2{}^2,$$

and $D_x{}^l u (D_t{}^l u)$ will stand for any partial derivative of u of order l $(D_x{}^0 u = D_t{}^0 u = u)$ with respect to the variables (x_1, x_2) $((t_1, t_2))$. Furthermore, the index in any sum will always vary successively over the positive integers. Finally, the notation "Theorem 12.2" refers to the second theorem of Section twelve.

2. Characteristic functions with estimates. The characteristic functions and values corresponding to the Laplacian operator and the region $G(X)$

are indicated by $\phi_n = \phi_n(x)$ and λ_n $(n = 1, 2, \cdots)$. They satisfy the following equations:

$$\Delta_x \phi_n + \lambda_n \phi_n = 0 \qquad x \in G(X), \tag{2.1}$$

$$\phi_n(x) = 0 \qquad x \in G^*(X), \tag{2.2}$$

$$\int_{G(X)} \phi_n^2 \, dx = 1. \tag{2.3}$$

In addition, the dependence of the maxima of the moduli of ϕ_n and its partial derivatives upon λ_n are known. Indeed, A. Hammerstein [6] has shown the existence of a positive constant C which is independent of x and n and for which

$$| D_x^k \phi_n | \leqq C \lambda_n^{1+k} \qquad (k = 0, 1, \text{ and } n = 1, 2, \cdots), \tag{2.4}$$

provided that $G^*(X)$ is formed from a finite number of analytic arcs. Furthermore, J. Schauder [7] has investigated the general linear second order elliptic differential equation and has determined bounds for the second and third order partial derivatives of the solution. It follows, as an easy consequence of the bounds of Schauder and those of Hammerstein, that

$$| D_x^k \phi_n | \leqq C \lambda_n^{1+k} \qquad (k = 2, 3, \text{ and } n = 1, 2, \cdots). \tag{2.5}$$

3. Statement of existence theorem.

THEOREM 3.1. *There exists a $u(x, t)$, with $D_x{}^l u$ and $D_t{}^l u$ $(l = 0, 1, 2)$ continuous on $\bar{G}(X) \times G(T)$, which simultaneously satisfies the ultrahyperbolic equation*

$$\Delta_t u = \Delta_x u \qquad (x, t) \in G, \tag{3.1}$$

the boundary condition

$$u(x, t) = 0 \qquad (x, t) \in G^*(X) \times G(T), \tag{3.2}$$

and the integral condition

$$\int_0^\infty u(x, t) \, ds = F(x, \theta) \qquad (t_1 + i t_2 = s e^{i\theta}), \tag{3.3}$$

the integral converging uniformly in (x, θ).

It is assumed that the arbitrarily prescribed $F(x, \theta)$ is of period 2π in θ and in addition possesses the following properties: (i) $\Delta_x{}^l F$ $(l = 0, 1, \cdots, 5)$ and $D_x{}^1(\Delta_x{}^l F)$ $(l = 0, 1, \cdots, 4)$ are continuous on

$$\bar{G}(X) \times I, \qquad I = \{\theta \mid 0 \leqq \theta \leqq 2\pi\}.$$

(ii) $\Delta_x{}^l F$ $(l=0,1,\cdots,5)$ vanishes at $G^*(X)$. (iii) the normal derivative $\partial\Delta_x F/\partial\nu$ $(l=0,1,\cdots,5)$ is bounded on $G^*(X)$. (iii) $F(x,\theta)$ and $\Delta_x{}^2 F$ satisfy a Hölder condition with respect to θ, uniformly in (x,θ), with Hölder exponent $\alpha>\frac{1}{2}$; that is, there is a constant H independent of x and θ such that

$$\begin{aligned} &|F(x,\theta_2)-F(x,\theta_1)|\leqq H|\theta_2-\theta_1|^\alpha, \\ &|\Delta_x{}^2F(x,\theta_2)-\Delta_x{}^2F(x,\theta_1)|\leqq H|\theta_2-\theta_1|^\alpha \end{aligned} \tag{3.4}$$

for $(x,\theta_i)\in G(X)\times I$, $i=1,2$.

4. Formal series representation of solutions. Assume that $u(x,t)$ is a solution of (0.1) which vanishes for $(x,t)\in G^*(X)\times G(T)$ and that $D_x{}^k u$ $(k=0,1,2)$ is continuous for $(x,t)\in\bar{G}(X)\times G(T)$. Then, as is well known (see [8], p. 369), $u(x,t)$ admits the series representation

$$u(x,t)=\sum\phi_n(x)\int_{G(X)}u(x,t)\phi_n(x)\,dx=\sum v_n(t)\phi_n(x), \tag{4.1}$$

where

$$v_n(t)=\int_{G(X)}u(x,t)\phi_n(x)\,dx. \tag{4.2}$$

Now, suppose that $u(x,t)$ fulfills (3.1)-(3.3) and $D_t{}^k u$ $(k=0,1,2)$ is continuous on $\bar{G}(X)\times G(T)$. Then, because of (4.2),

$$\begin{aligned} \Delta_t v_n &= \int_{G(X)}\phi_n\Delta_x u\,dx \\ &= \int_{G(X)}u\Delta_x\phi_n\,dx+\int_{G^*(X)}[\phi_n(\partial u/\partial\nu)-u(\partial\phi_n/\partial\nu)]\,dG^*; \end{aligned} \tag{4.3}$$

where $\partial u/\partial\nu$ and $\partial\phi_n/\partial\nu$ denote the normal derivatives of u and ϕ_n at $G^*(X)$. Since $\partial\phi_n/\partial\nu$ $(n=1,2,\cdots)$ are uniformly bounded at $G^*(X)$, see (2.4) and (2.5), the line integral appearing in (4.3) vanishes. Thus $v_n(t)$ must be a solution of the reduced wave equation

$$\Delta_t v_n+\lambda_n v_n=0 \qquad t\in G(t). \tag{4.4}$$

Besides, by reason of (3.3) and (4.2), $v_n(t)$ will also satisfy the integral condition

$$\int_0^\infty v_n(t)\,ds=\int_{G(X)}F(x,\theta)\phi_n\,dx\equiv F_n(\theta). \tag{4.5}$$

Hence $v_n(t)$ simultaneously satisfies (4.4) and (4.5). Conversely, by Theorem 5.1, the conditions (4.4) and (4.5) assure the existence of a unique $v_n(t)$. Consequently, this completes the demonstration that the series (4.1)

represents the solution of the indicated problem provided that the series and its derivatives can be properly majorized and provided that (4.1) uniformly satisfies the integral condition (3.3).

5. The reduced wave equation. The following result was recently established in [9].

THEOREM 5.1. *There exists a unique $v(t)$, with $D_t^l v$ $(l=0,1,2)$ continuous on $G(T)$, which simultaneously satisfies the reduced wave equation*

$$\Delta_t v + \lambda v = 0 \qquad t \in G(T), \tag{5.1}$$

λ a positive constant, and the integral condition

$$\int_0^\infty v(t)\,ds = f(\theta) \qquad (t_1 + it_2 = se^{i\theta}), \tag{5.2}$$

the integral converging uniformly in θ.

It is assumed that the arbitrarily prescribed $f(\theta)$ is of period 2π in θ and in addition it posesses the following property: $f(\theta)$ fulfills a uniform Hölder condition with Hölder exponent $\alpha > \frac{1}{2}$; that is, there is a constant H independent of θ such that

$$|f(\theta_2) - f(\theta_1)| \leqq H\,|\theta_2 - \theta_1|^\alpha \tag{5.3}$$

for $\theta_i \in I$, $i = 1, 2$.

Finally, the solution $v(t)$ has the following explicit representation:

$$2\pi^2 v(t) = \pi\lambda^{\frac{1}{2}} \int_0^{2\pi} f(\phi) K_1(\lambda^{\frac{1}{2}} s, \phi - \theta)\,d\theta \tag{5.4}$$

$$+ \lambda^{\frac{1}{2}} \int_0^{2\pi} d\phi f(\phi) \int_0^{\pi/2} (\sin\tau)^{-1} [K_2(\lambda^{\frac{1}{2}} s, \tau + \phi - \theta) + K_2(\lambda^{\frac{1}{2}} s, \tau - \phi + \theta)]\,d\tau,$$

where

$$K_1(s,\theta) = \cos[s\sin\theta], \qquad K_2(s,\theta) = \sin[s\sin\theta]. \tag{5.5}$$

By Theorem 5.1 the function v_n satisfying (4.4) and (4.5) is uniquely and explicitly given by the formula (5.4) with λ replaced by λ_n and $f(\phi)$ by $F_n(\phi)$. The fact that $f(\phi) \equiv F_n(\phi)$ satisfies the Hölder condition (5.3) is shown in the proof of Lemma 11.1, and is a simple consequence of fact that $F(x,\theta)$ satisfies the Hölder condition (3.4). Hence, the formal series for $u(x,t)$ is

$$u(x,t) = \sum v_n(t)\phi_n(x), \tag{5.6}$$

where

$$2\pi^2 v_n(t) = \pi\lambda_n^{\frac{1}{2}} \int_0^{2\pi} F_n(\phi) K_1(\lambda_n^{\frac{1}{2}} s, \phi - \theta)\, d\phi$$

$$+ \lambda_n^{\frac{1}{2}} \int_0^{2\pi} d\phi F_n(\phi) \int_0^{\pi/2} (\sin\tau)^{-1} \tag{5.7}$$

$$\times [K_2(\lambda_n^{\frac{1}{2}} s, \tau + \phi - \theta) + K_2(\lambda_n^{\frac{1}{2}} s, \tau - \phi + \theta)]\, d\tau$$

and

$$F_n(\theta) = \int_{G(X)} F(x, \theta)\phi_n(x)\, dx. \tag{5.8}$$

6. A bound for F_n.

LEMMA 6.1. *Assume that* $\partial \Delta_x{}^l F/\partial \nu$ $(l = 0, \cdots, k-1)$ *is bounded on* $G^*(X)$ *and* $\Delta_x{}^l F(x, \theta)$ $(l = 0, \cdots, k-1)$ *vanishes at* $G^*(X)$. *Then*

$$F_n(\theta) = (-1)^k \lambda_n{}^{-k} \int_{G(X)} \phi_n \Delta_x{}^k F\, dx. \tag{6.1}$$

Proof. The relation (6.1) follows by $k-1$ applications of Green's formula.

COROLLARY 6.1. *Retaining the hypotheses of Lemma* 6.1,

$$|F_n(\theta)| \leqq B_k \lambda_n{}^{-k}, \text{ where } B_k{}^2 = \operatorname*{Max}_{\theta \in I} \int_{G(X)} [\Delta_x{}^k F]^2\, dx. \tag{6.2}$$

Proof. Applying Schwarz's inequality to (6.1) and making use of (2.3) yields (6.2).

7. A bound for v_n.

LEMMA 7.1. *Set* $\mu = \operatorname*{Max}_{n=1,2,\cdots} \lambda_n{}^{-\frac{1}{2}}$. *Then, under the hypotheses of Lemma* 6.1,

$$|v_n(t)| \leqq B_k \lambda_n{}^{1-k}(s + \mu). \tag{7.1}$$

Proof. Define $v_1 = v_1(t, n)$ and $v_2 = v_2(t, n)$ by the following two equations:

$$2\pi\lambda_n{}^{-\frac{1}{2}} v_1 = \int_0^{2\pi} F_n(\phi) K_1(\lambda_n{}^{\frac{1}{2}} s, \phi - \theta)\, d\phi, \tag{7.2}$$

$$2\pi\lambda_n{}^{-\frac{1}{2}} v_2 = \pi^{-1} \int_0^{2\pi} d\phi F_n(\phi) \int_0^{\pi/2} (\sin\tau)^{-1} [K_2(\lambda_n{}^{\frac{1}{2}} s, \tau + \phi - \theta) \tag{7.3}$$

$$+ K_2(\lambda_n{}^{\frac{1}{2}} s, \tau - \phi + \theta)]\, d\tau.$$

Therefore,

$$v_n(t) = v_1(t, n) + v_2(t, n); \tag{7.4}$$

so, an estimate for v_n will be obtained as soon as estimates for v_1 and v_2 are known. A bound for v_1 is immediately found by reason of (6.2) and (7.2), namely,

$$|v_1(t,n)| \leqq \lambda_n^{\frac{1}{3}} \operatorname{Max} |F_n(\phi)| \leqq B_k \lambda_n^{\frac{1}{3}-k}. \tag{7.5}$$

An estimate for v_2 will now be derived. Set $\delta = \phi - \theta$ and

$$I(s,\delta) = \int_0^{\pi/2} (\sin\tau)^{-1}[K_2(s,\tau+\delta) - K_2(s,\delta-\tau)]\,d\tau. \tag{7.6}$$

Hence, because of (6.2), (7.3) and (7.6),

$$\pi |v_2(t,n)| \leqq B_k \lambda_n^{\frac{1}{3}-k} \operatorname{Max} |I(\lambda_n^{\frac{1}{3}}s,\delta)|. \tag{7.7}$$

Moreover, because of the mean value theorem,

$$\begin{aligned} |K_2(s,\tau+\delta) - K_2(s,\delta-\tau)| &\leqq s\,|\sin(\tau+\delta) - \sin(\delta-\tau)| \\ &= 2s\,|\sin\tau\cos\delta| \end{aligned} \tag{7.8}$$

which implies

$$|I(s,\delta)| \leqq 2s\int_0^{\pi/2} d\tau \leqq \pi s. \tag{7.9}$$

It thus will follow from (7.7) and (7.9) that

$$|v_2(t,n)| \leqq s B_k \lambda_n^{1-k}. \tag{7.10}$$

Consequently, the bound (7.1) for v_n follows from the inequalities (7.5) and (7.10).

8. Bounds for the derivatives of v_1. By defining

$$\alpha_2 + i\alpha_1 = -\exp[-i\phi] \tag{8.1}$$

and recalling that $t_1 + it_2 = s\exp[i\theta]$, one finds that

$$K_1(\lambda_n^{\frac{1}{3}}s, \phi-\theta) = \cos[\lambda_n^{\frac{1}{3}}(\alpha_1 t_1 + \alpha_2 t_2)]. \tag{8.2}$$

Hence, using (7.2),

$$v_1 = \tfrac{1}{2\pi}\int_0^{2\pi} \lambda_n^{\frac{1}{3}} F_n(\phi)\cos[\lambda_n^{\frac{1}{3}}(\alpha_1 t_1 + \alpha_2 t_2)]\,d\phi \tag{8.3}$$

which implies that

$$\partial v_1/\partial t_i = -\tfrac{1}{2\pi}\int_0^{2\pi} \alpha_i \lambda_n F_n(\phi)\sin[\lambda_n^{\frac{1}{3}}(\alpha_1 t_1 + \alpha_2 t_2)]\,d\phi \tag{8.4}$$

and

$$\partial^2 v_1/\partial t_i \partial t_j = -\tfrac{1}{2\pi}\int_0^{2\pi} \alpha_i\alpha_j \lambda_n^{\frac{2}{3}} F_n(\phi)\cos[\lambda_n^{\frac{1}{3}}(\alpha_1 t_1 + \alpha_2 t_2)]\,d\phi. \tag{8.5}$$

From the last two equations, together with (6.2), one deduces the following estimates:

$$|\partial v_1/\partial t_i| \leqq B_k \lambda_n^{1-k} \tag{8.6}$$

and

$$|\partial^2 v_1/\partial t_i \partial t_j| \leqq B_k \lambda_n^{\frac{3}{2}-k}. \tag{8.7}$$

9. Bounds for the derivatives of v_2. By defining

$$\beta_2 + i\beta_1 = -\exp[-i(\tau+\phi)], \qquad \gamma_2 + i\gamma_1 = \exp[i(\tau-\phi)] \tag{9.1}$$

and placing

$$\Omega(t,\phi,\tau) \equiv K_2(\lambda_n^{\frac{1}{3}}s, \tau+\phi-\theta) + K_2(\lambda_n^{\frac{1}{3}}s, \tau-\phi+\theta), \tag{9.2}$$

one finds that

$$\Omega = \sin[\lambda_n^{\frac{1}{3}}(\beta_1 t_1 + \beta_2 t_2)] + \sin[\lambda_n^{\frac{1}{3}}(\gamma_1 t_1 + \gamma_2 t_2)]. \tag{9.3}$$

Hence, by (7.3),

$$v_2 = \tfrac{1}{2\pi^2}\int_0^{2\pi} d\phi\, \lambda_n^{\frac{1}{3}} F_n(\phi) \int_0^{\pi/2} \Omega(t,\phi,\tau)\,(\sin\tau)^{-1}\, d\tau; \tag{9.4}$$

and, by (9.3),

$$\begin{aligned}\partial\Omega/\partial t_i = {} & \lambda_n^{\frac{1}{3}}\beta_i \cos[\lambda_n^{\frac{1}{3}}(\beta_1 t_1 + \beta_2 t_2)] \\ & + \lambda_n^{\frac{1}{3}}\gamma_i \cos[\lambda_n^{\frac{1}{3}}(\gamma_1 t_1 + \gamma_2 t_2)],\end{aligned} \tag{9.5}$$

$$\begin{aligned}\partial^2\Omega/\partial t_i \partial t_j = {} & -\lambda_n \beta_i\beta_j \sin[\lambda_n^{\frac{1}{3}}(\beta_1 t_1 + \beta_2 t_2)] \\ & - \lambda_n \gamma_i\gamma_j \sin[\lambda_n^{\frac{1}{3}}(\gamma_1 t_1 + \gamma_2 t_2)].\end{aligned} \tag{9.6}$$

From (9.5) and (9.6) one can derive estimates for the derivatives with respect to τ of $\partial\Omega/\partial t_i$ and $\partial^2\Omega/\partial t_i\partial t_j$. Because of these estimates and the fact that the left-members of (9.5) and (9.6) vanish for $\tau = 0$, it follows from the mean value theorem that

$$|\partial\Omega/\partial t_i| \leqq 2\lambda_n \tau(s+\mu) \tag{9.7}$$

and

$$|\partial^2\Omega/\partial t_i\partial t_j| \leqq 2\lambda_n^{\frac{2}{3}}(s+2\mu), \tag{9.8}$$

where $\mu = \operatorname{Max}_{n=1,2,\cdots} \lambda_n^{-\frac{1}{3}}$.

It is now possible to find estimates for the first and second partial derivatives of $v_2(t,n)$. First, these derivatives are given by the formulas

$$\partial v_2/\partial t_i = \tfrac{1}{2\pi^2}\int_0^{2\pi} d\phi \lambda_n^{\frac{1}{3}} F_n(\phi) \int_0^{\pi/2} (\sin\tau)^{-1}(\partial\Omega/\partial t_i)\, d\tau \tag{9.9}$$

and

$$(9.10)\qquad \partial^2 v_2/\partial t_i \partial t_j = \tfrac{1}{2\pi^2}\int_0^{2\pi} d\phi \lambda_n^{\frac{1}{2}} F_n(\phi) \int_0^{\pi/2} (\sin\tau)^{-1}(\partial\Omega/\partial t_i \partial t_j)\, d\tau.$$

Thus, by (6.2), (9.7) and (9.9),

$$(9.11)\qquad |\partial v_2/\partial t_i| \leqq 2(s+\mu) B_k \lambda_n^{\frac{3}{2}-k} \int_0^{\pi/2} \tau(\sin\tau)^{-1} d\tau \leqq 2(s+\mu) C B_k \lambda_n^{\frac{3}{2}-k}$$

and, by (6.2), (9.8) and (9.10),

$$(9.12)\qquad |\partial^2 v_2/\partial t_i \partial t_j| \leqq 2(s+\mu) B_k \lambda_n^{2-k} \int_0^{\pi/2} \tau(\sin\tau)^{-1} d\tau \leqq 2(s+\mu) C B_k \lambda_n^{2-k},$$

with a suitably chosen constant C.

As $v_n(t) = v_1(t,n) + v_2(t,n)$, the inequalities (8.6) and (9.11) yield

$$(9.13)\qquad |\partial v_n/\partial t_i| \leqq B_k \lambda_n^{\frac{3}{2}-k}[\mu + 2C(s+\mu)],$$

while (8.7) and (9.12) yield

$$(9.14)\qquad |\partial^2 v_n/\partial t_i \partial t_j| \leqq B_k \lambda_n^{2-k}[\mu + 2C(s+\mu)].$$

10. Majorization of series. The solution $u(x,t)$ has been defined by the formal series

$$(10.1)\qquad u(x,t) = \sum v_n(t)\phi_n(x),$$

where $v_n(t)$ is uniquely determined by means of the conditions (4.4) and (4.5). The series (10.1), together with its formal partial derivatives, can be majorized by making use of the estimates given in Sections two, seven and nine. In particular, the bounds (2.4), (7.1) and (9.13) give the following majorizations:

$$(10.2)\qquad \sum |v_n(t)\phi_n(x)| \leqq C B_k (s+\mu) \sum \lambda_n^{2-k},$$

$$(10.3)\qquad \sum |v_n D_x^1 \phi_n| \leqq C B_k (s+\mu) \sum \lambda_n^{3-k}$$

$$(10.4)\qquad \sum |D_t^1 v_n \phi_n| \leqq C B_k (\mu + 2C(s+\mu)) \sum \lambda_n^{\frac{5}{2}-k},$$

while the bounds (2.4), (2.5), (7.1) and (9.14) give

$$(10.5)\qquad \sum |v_n D_x^2 \phi_n| \leqq C B_k (s+\mu) \sum \lambda_n^{4-k}$$

$$(10.6)\qquad \sum |D_t^2 v_n \phi_n| \leqq C B_k (\mu + 2C(s+\mu)) \sum \lambda_n^{3-k}.$$

By reason of the well-known inequality, $\sum \lambda_n^{-2} < \infty$ (see [8], p. 130), and the above estimates, the following lemma and corollary have been established.

LEMMA 10.1. *The series* (10.2)-(10.6) *are uniformly convergent, for* $k=6$, *on the closure of* $G(X)\times G_N(T)$, *where* N *is any positive real number and* $G_N(T)=\{(t_1,t_2)\,|\,-N\leqq t_1,t_2\leqq N\}$.

COROLLARY 10.1. $u(x,t)$ *of* (10.1) *and the partial derivatives* $D_t{}^l u$, $D_x{}^l u$ $(l=1,2)$ *exist and are continuous on the closure of* $G(X)\times G_N(T)$. *Furthermore,* $u(x,t)$ *is a solution of the ultrahyperbolic equation* (0.1).

By reason of Corollary 10.1, the proof of Theorem 3.1 has now been completed except for the fact that the integral condition (3.3) has still to be established.

11. Verification of integral condition. Here in this section it will be shown that $u(x,t)$ is such that

$$\int_0^\infty u(x,t)\,ds=F(x,\theta), \tag{11.1}$$

the integral converging uniformly with respect to $(x,\theta)\in G(X)\times I$. Since the series (10.1) is uniformly convergent on $G(X)\times G_N(T)$,

$$\int_0^N u(x,t)\,ds=\sum\phi_n(x)\int_0^N v_n(t)\,ds. \tag{11.2}$$

In order to discuss the passage to the limit $(N\to\infty)$ in (11.2), it is necessary to use the following lemma.

LEMMA 11.1. *The integral*

$$\int_0^\infty v_n(t)\,ds \qquad (n=1,2,\cdots) \tag{11.3}$$

converges uniformly with respect to (n,θ).

Proof. It was proved in the paper [9] that the integral (11.3) converges uniformly in θ when n is fixed. That the convergence is uniform with respect to (n,θ) will now be established. First, the integrals (3.7) and (3.8) of paper [9], page 394, converge to zero uniformly with respect to (n,θ) provided that the Hölder coefficients A_n of F_n are uniformly bounded and the Hölder exponents have a greatest lower bound which is greater than $\frac{1}{2}$. But this is clear from (5.8), by virtue of Schwarz's inequality, the normalization condition (2.3), and the Hölder condition, (3.4), satisfied by F. Furthermore, because of the assertion, the integral (4.3) of [9], p. 394, converges uniformly in (n,θ). Consequently, the condition (3.4) of the present paper is sufficient to assure that the integrals (11.3) converge uniformly in (n,θ).

It is now possible to investigate the passage to the limit, $N \to \infty$, in equation (11.2).

LEMMA 11.2. *The $u(x,t)$ of (10.1) is such that the integral relation (11.1) holds uniformly in (x,θ).*

Proof. Since $F(x,\theta)$ is subject to the conditions of Theorem 3.1, it can be expanded into the series

$$F(x,\theta) = \sum F_n(\theta)\phi_n(x), \tag{11.4}$$

where $F_n(\theta)$ is defined by equation (5.8). Thus,

$$\int_0^N u(x,t)\,ds - F(x,\theta) = \sum \Big[\int_0^N v_n(t)\,ds - F_n(\theta) \Big]\phi_n(x). \tag{11.5}$$

Now, setting

$$F_{nk}(\theta) \equiv \int_{G(X)} \phi_n(x)\,\Delta_x{}^k F(x,\theta)\,dx, \tag{11.6}$$

the Lemma 6.1 asserts that

$$F_n(\theta) = (-1)^k \lambda_n{}^{-k} F_{nk}(\theta), \tag{11.7}$$

with k to be presently fixed. Assuming that $\Delta_x{}^k(x,\theta)$ is such that

$$|\,\Delta_x{}^k F(x,\theta_2) - \Delta_x{}^k F(x,\theta_1)\,| < H\,|\,\theta_2 - \theta_1\,|^\alpha \qquad (\alpha > \tfrac{1}{2}), \tag{11.8}$$

H an absolute constant, it follows that $F_{nk}(\theta)$ will satisfy the same type of Hölder condition with Hölder coefficient uniformly bounded. Hence, if $v_{nk} = (-1)^k \lambda_n{}^k v_n$, then $\Delta_t v_{nk} + \lambda_n v_{nk} = 0$ and

$$\int_0^\infty v_{nk}(t)\,ds = F_{nk}(\theta), \tag{11.9}$$

the integral converging uniformly in (n,θ). Thus, the equation (11.5) can be put in the form

$$\int_0^N u(x,t)\,ds - F(x,\theta) = \sum (-1)^k \lambda_n{}^{-k} \Big[\int_0^N v_{nk}(t)\,ds - F_{nk}(\theta) \Big]\phi_n(x). \tag{11.10}$$

Furthermore, given an $\epsilon > 0$, the uniform convergence of (11.9) implies the existence of an $N_0(\epsilon)$, independent of (n,θ), for which

$$\Big|\, \int_0^N v_{nk}(t)\,ds - F_{nk}(\theta) \,\Big| < \epsilon \tag{11.11}$$

for all $N \geqq N_0(\epsilon)$. Consequently, (11.10) and (11.11) yield the next inequality.

$$(11.12)\qquad \Big|\int_0^N u(x,t)\,ds - F(x,\theta)\Big| \leqq \epsilon\sum \lambda_n^{-k}\,|\phi_n(x)| \leqq \epsilon[\sum \lambda_n^{2(1-k)}]^{\frac{1}{2}}\cdot[\sum \phi_n^2(x)\lambda_n^{-2}]^{\frac{1}{2}}.$$

As the factor of ϵ in the last term in uniformly bounded for $k=2$ (see [8], p. 130) and as (11.8) holds by hypothesis (see Theorem 3.1) for $k=2$, the inequality (11.2) implies the validity of (11.1).

12. Uniqueness of solution and conditional convergence of integral.

THEOREM 12.1. *Assume there exists a $u(x,t)$, with $D_x^l u$ and $D_t^l u$ $(l=0,1,2)$ continuous on $\bar{G}(X)\times G(T)$, which simultaneously satisfies the ultrahyperbolic equation*

$$(12.1)\qquad \Delta_t u = \Delta_x u \qquad (x,t)\in G,$$

the boundary condition

$$(12.2)\qquad u(x,t)=0 \qquad (x,t)\in G^*(X)\times G(T),$$

and the integral condition

$$(12.3)\qquad \int_0^\infty u(x,t)\,ds = 0,$$

the integral converging uniformly in (x,θ). Then $u(x,t)$ is necessarily identically zero.

Proof. Because of the boundary condition and the differentiability conditions imposed on $u(x,t)$, it follows that it has the series representation (4.1) where $v_n(t)$ is given by (4.2). Therefore, by the uniformity of (12.3),

$$(12.4)\qquad \int_0^\infty v_n(t)\,ds = \int_{G(X)} dx\,\phi_n(x)\int_0^\infty u(x,t)\,ds = 0.$$

By Theorem 5.1 the only solution of (4.4) subject to (12.4) is $v_n(t)\equiv 0$ $(n=1,2,\cdots)$. Hence, because of the series representation (4.1), $u(x,t)$ is identically zero on G. That is, the solution of Theorem 3.1 is unique.

The fact that the integral (3.3) is only conditionally convergent is the context of the next theorem.

THEOREM 12.2. *Assume that $u(x,t)$ satisfies all of the hypotheses of Theorem 12.1 except that (12.3) is replaced by the requirement that the integral*

$$(12.5)\qquad \int_0^\infty |u(x,t)|\,ds$$

be uniformly convergent with respect to $(x,\theta) \in G(X) \times I$. *Then* $u(x,t)$ *is necessarily identically zero.*

Proof. On account of (4.2)

$$|v_n(t)| \leqq \int_{G(X)} |u(x,t)|\,|\phi_n(x)|\,dx, \tag{12.6}$$

and this implies that

$$\int_0^N |v_n(t)|\,ds \leqq \int_{G(X)} dx\,|\phi_n(x)| \int_0^N |u(x,t)|\,ds. \tag{12.7}$$

Thus, because of the uniform convergence of (12.5),

$$\int_0^\infty |v_n(t)|\,dt \leqq \int_{G(X)} dx\,|\phi_n(x)| \int_0^\infty |u(x,t)|\,ds < \infty, \tag{12.8}$$

the convergence being uniform in θ. Hence, by Corollary 3 of [9], p. 390, $v_n(t) \equiv 0$ $(n = 1, 2, \cdots)$ on $G(T)$. Consequently, because of the series representation (4.1), $u(x,t)$ is identically zero on G.

WAYNE STATE UNIVERSITY.

REFERENCES.

[1] G. Hamel, "Über die Geometrieen in denen die Geraden die Kürzesten sind," *Mathematischen Annalen*, vol. 57 (1903), pp. 231-264.

[2] F. John, "The ultrahyperbolic differential equation with four independent variables," *Duke Mathematical Journal*, vol. 4 (1938), pp. 300-322.

[3] N. S. Piskunov, "On the characteristic problem for equations of ultrahyperbolic type," *Doklady Akademie Nauk SSSR*, vol. 59 (1948), pp. 439-442.

[4] J. Hadamard, *Lectures on Cauchy's Problem*, Yale University Press, 1923, pp. 4 and 39.

[5] I. G. Petrowsky, "On some problems of the theory of partial differential equations," *American Mathematical Society Translations*, Number 12.

[6] A. Hammerstein, "Über Entwicklung gegebener Funktionen nach Eigenfunktionen von Randwertaufgaben," *Mathematische Zeitschrift*, vol. 27 (1927), pp. 269-311, especially pp. 290-294.

[7] J. Schauder, "Über den Zusammenhang zwischen der Eindeutigkeit und Lösbarkeit partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus," *Mathematische Annalen*, vol. 106 (1932), pp. 661-721, especially pp. 687-688.

[8] R. Courant and D. Hilbert, Methods of mathematical physics, vol. 1, New York, 1953.

[9] O. G. Owens, "A characterization of the everywhere regular solution of the reduced wave equation," *Transactions of the American Mathematical Society*, vol. 88 (1958), pp. 388-399.

QUADRATIC FORMS OVER FIELDS OF CHARACTERISTIC 2.*

By CHIH-HAN SAH.

In [13], Minkowski solved the problem of rational equivalence for quadratic forms. This was extended by Hasse to algebraic number fields, [10], [11]. Witt [16], and Arf [1], then developed general theories of quadratic forms over fields of characteristic $\neq 2$ and $= 2$ respectively. These works together solved the field equivalence problem of quadratic forms over sufficiently "nice" fields, e. g. global and local fields.

The problem of integral equivalence for quadratic forms over a global field remains open. In a sequence of papers, O'Meara [14], [15], solved the problem of integral equivalence for quadratic forms over a local field of characteristic $\neq 2$. Earlier, Arf [2], solved the same problem for binary and ternary quadratic forms over local fields of characteristic 2. The purpose of the present investigation, which formed the basic part of the author's thesis, is to give a solution to the integral equivalence problem for quadratic forms in any finite number of variables over a local field of characteristic 2. The main results of this paper are contained in Theorem 4. 6, for modular lattices, and in Theorem 5. 5 for lattices in general.

Familiarity of [1] is assumed. Basic notions on quadratic forms may be found in [4], [5] and [9]. Basic results on fields and algebras may be found in [3], [5] and [17].

I wish to thank Prof. O'Meara for introducing me to the theory of quadratic forms in his seminar given during the spring of 1958 as well as the many interesting conversations on quadratic forms during the preparation of this paper.

1. Notations. Throughout this paper, Ω shall denote a finite field of characteristic 2. $k = \Omega\langle\langle\pi\rangle\rangle$, the formal power-series field over Ω with uniformizer π, $\pi^\infty = 0$. $\mathfrak{o} = \Omega[[\pi]]$, the ring of integral power series. $\Omega[\pi^{-1}]$, the polynomial ring in π^{-1} over Ω. 0 and λ are fixed representatives of $\Omega/\mathfrak{P}\Omega$, where $\mathfrak{P}x = x^2 + x$. ord is the ordinal function in k and Ord is the ordinal function in $k/\mathfrak{P}k$ defined by $\mathrm{Ord}(\Delta) = \max\{\mathrm{ord}(x) \mid x \in \Delta\}$, where $\Delta \in k/\mathfrak{P}k$.

* Received November 24, 1959.

By a quadratic space (respectively lattice) U over k, we mean a vector space (respectively free module) of finite type over k (respectively $\mathfrak{o}$), together with a map $Q: U \to k$ such that $Q(ax) = a^2 Q(x)$ for $a \in k$ (respectively $a \in \mathfrak{o}$) and $x \in U$, and such that $\langle x, y\rangle = Q(x+y) + Q(x) + Q(y)$ is bilinear. If U is the direct sum of U_1 and U_2 and $\langle U_1, U_2\rangle = 0$, then we shall write $U = U_1 \oplus U_2$; such decompositions will be called orthogonal. $\dim U =$ rank of U over k (or $\mathfrak{o}$). $R(U) = \{x \in U \mid Q(x) = 0, \langle x, U\rangle = 0\}$ and $K(U) = \{x \in U \mid \langle x, U\rangle = 0\}$ will be called the radical and core of U respectively. It is easily seen that $R(U)$ and $K(U)$ are orthogonal summands of U, and that the structure of U is uniquely determined by $R(U)$ and the natural quadratic structure on $U/R(U)$. Thus, we shall always assume that the given quadratic spaces or quadratic lattices are non-degenerate, i.e. $R(U) = 0$. U is called non-defective when $K(U) = 0$. It is easy to see that $\dim K(U) \leqq [k : k^2] = 2$.

If M is a quadratic lattice, then $k \otimes_{\mathfrak{o}} M$ receives a natural quadratic structure, this quadratic space shall be denoted by kM. If $a \neq 0 \in k$, then the space (or lattice) M together with the quadratic map $a \circ Q$ defined by $(a \circ Q)(x) = aQ(x)$ for $x \in M$, shall be denoted by $a \circ M$. This operation is called scaling by a. Homomorphisms which preserve the quadratic structures will be called representations. Isomorphic representations are called isometries and denoted by $\cong$. From the non-degeneracy hypothesis, it follows easily that representations are always monomorphic.

The quadratic lattice $\mathfrak{o}x$ such that $Q(x) = a \neq 0 \in k$ will be denoted by (a). The quadratic lattice $\mathfrak{o}x + \mathfrak{o}y$ such that $Q(x) = a$, $\langle x, y\rangle = b \neq 0$ and $Q(y) = c$, where $a, b, c \in k$, shall be denoted by $\begin{pmatrix} b \\ a & c \end{pmatrix}$. The subscript k will be used to denote the quadratic spaces spanned by the corresponding lattices. If the basis element x is to be emphasized, we shall write $\mathfrak{o}x = (a)$, otherwise, we shall use $\mathfrak{o}x \cong (a)$, similarly for binary and higher dimensional lattices and spaces.

If V is a quadratic space, then we shall let $C(V)$ denote the Clifford algebra of V, cf. [5; Chap. 9], [9]. If V is non-defective, then $C(V)$ is a k-central simple algebra and $C((1)_k \oplus V)/\mathfrak{N} \approx C(V)$, where $\mathfrak{N}$ is the radical of the algebra $C((1)_k \oplus V)$, cf. [1; p. 151]. In general, $C(U \oplus V)$ is isomorphic to $C(U) \oplus_k C(V)$. If $V = \begin{pmatrix} 1 \\ a & c \end{pmatrix}_k$, then the class of $C(V)$ in the Brauer group of k is denoted by $[a, c]$. It can be shown, [17; p. 135] that, $[a, c] = [c, a]$, $[ad^2, c] = [a, cd^2]$, $[d^2, c] = 1$ and $[a, b+c] = [a, b][a, c]$, where the group operation in the Brauer group is written multiplicatively. Using these, it is easy to see that $[a, c] = 1$ or $[\pi^{-1}, \pi\lambda]$ $(\neq 1)$.

If $V \cong \oplus \begin{pmatrix} b_i \\ a_i \quad c_i \end{pmatrix}_k$ is a non-defective quadratic space, then the Arf invariant is denoted by $\Delta(V) = \sum a_i c_i b_i^{-2} + \mathfrak{P}k \in k/\mathfrak{P}k$, cf. [1], [6], [12] or [18]. We shall need the following fact:

LEMMA 1.1. *Let* $\Delta \in k/\mathfrak{P}k$ *and* $a \in k$, *then,*

1). $\mathrm{Ord}(\Delta) > 0$ *implies that* $\Delta = \mathfrak{P}k$ *and* $\mathrm{Ord}(\Delta) = \infty$;

2). $\mathrm{Ord}(\Delta) = 0$ *implies that* $\Delta = \lambda + \mathfrak{P}k$ *and conversely*;

3). $\mathrm{Ord}(\Delta) < 0$ *implies that* $\mathrm{Ord}(\Delta) = 2s+1$, s *an integer*;

4). $a \in \Delta$ *and* $\mathrm{ord}(a) = 2s+1 < 0$ *implies that* $\mathrm{Ord}(\Delta) = 2s+1$;

5). $a = \pi^{-1}b^2 + D + \mathfrak{P}c$, *where* $b \in \Omega[\pi^{-1}]$ *and* $D = 0$ *or* λ; b, D *and* $\mathfrak{P}c$ *are uniquely determined by* a. *In particular, if* $\mathrm{ord}(a) = 2s+1 < 0$, *then* $\mathrm{ord}(\pi^{-1}b^2 + D) = \mathrm{ord}(\pi^{-1}b^2) = 2s+1 < \mathrm{ord}(\mathfrak{P}c)$; *if* $\mathrm{ord}(a) = 2s < 0$, *then* $\mathrm{ord}(\mathfrak{P}c) = 2s < \mathrm{ord}(\pi^{-1}b^2 + D)$; *if* $\mathrm{ord}(a) = 0$, *then* $b = 0$; *and, if* $\mathrm{ord}(a) > 0$, *then* $b = D = 0$.

Proof. Applying Hensel's Lemma [7; p. 43] to $X^2 + X + d$, where $d \in \mathfrak{o}$, we get 1) and 2) immediately.

3). Let $a \in \Delta$ such that $\mathrm{ord}(a) = \mathrm{Ord}(\Delta) < 0$. Suppose that $\mathrm{ord}(a) = 2s$ and $a = \pi^{2s}(b^2 + \pi c^2)$, where $s < 0$, $\mathrm{ord}(b) = 0$ and $\mathrm{ord}(c) \geqq 0$. Then $a' = a + \mathfrak{P}(\pi^s b) \in \Delta$ and $\mathrm{ord}(a') > 2s$, a contradiction.

4). Let $a' \in \Delta$ such that $\mathrm{ord}(a) = 2s+1 < \mathrm{ord}(a')$. Then, $a + a' \in \mathfrak{P}k$ and $\mathrm{ord}(a + a') = 2s+1 < 0$. Thus, $a + a' = b^2 + b$ and $\mathrm{ord}(b) < 0$. Therefore, $\mathrm{ord}(b^2 + b) = 2 \cdot \mathrm{ord}(b) = 2s+1$, a contradiction.

5) follows easily from the first four results. Q. E. D.

Let L be a quadratic lattice. $s(L) = \{\langle x, y\rangle \mid x, y \in L\}$ is called the scale of L. $Q(L) = \{Q(x) \mid x \in L\}$ is called the norm of L. $q(L) = \{Q(x) + s(L) \mid x \in L\}$ is called the norm group of L. For $-\infty < i \leqq \infty$, $L(i) = \{x \in L \mid \langle x, L\rangle \subseteq \pi^i \mathfrak{o}\}$ is called the i-th invariant sublattice of L and $q_i(L) = \{Q(x) + \pi^i \mathfrak{o} \mid x \in L(i)\}$ is called the i-th norm group of L.

It is easily seen that the scale is an $\mathfrak{o}$-module of finite type in k and that the norm groups are $\mathfrak{o}^2$-modules of finite type in k. If A, B are subsets of k, then let $A + B = \{a + b \mid a \in A, b \in B\}$. If $L = L_1 \oplus L_2$, then $s(a \circ L) = a \cdot s(L_1) + a \cdot s(L_2)$, similarly for $Q(a \circ L)$, $q(a \circ L)$ and $q_i(a \circ L)$. If $s(L) = \pi^i \mathfrak{o}$, then $L = L(i)$. In general $L \supseteq \cdots \supseteq L(i) \supseteq L(i+1) \supseteq \cdots \supseteq L(\infty) = K(L)$; $\pi^j L(i) \subseteq L(i+j)$; $kL(i) = kL$, when $i < \infty$; and $(L_1 \oplus L_2)(i) = L_1(i) \oplus L_2(i)$.

LEMMA 1.2. *Let L be a quadratic lattice. Then there exist $e \in k$ and $u \leqq v \leqq \infty$ such that $q(L) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$. Moreover, they may be chosen to satisfy the following:*

1). $\mathrm{ord}(e) = 0$ *and* $e\pi^u \in Q(L)$,

2). *If* $q(L) = 0$, *then* $u = v = \infty$,

3). *If* $\mathrm{rank}_{\mathfrak{o}^2} q(L) = 1$, *then* $u < v = \infty$, *and* $s(L) = 0$,

4). *If* $\mathrm{rank}_{\mathfrak{o}^2} q(L) = 2$, *then* $u < v < \infty$, $u + v \equiv 1 \bmod 2$ *and* $v \leqq s + 1$, *where* $s(L) = \pi^s \mathfrak{o}$.

When these conditions hold, then u and v are uniquely determined by L. Such representations of $q(L)$ will be called standard.

Proof. We may assume that $q(L) \neq 0$. Thus, $s(L) = \pi^s \mathfrak{o} \subseteq q(L)\mathfrak{o} = \pi^u \mathfrak{o}$. If $u < s$ $(s < \infty)$, then there exists $x \in L$ such that $\mathrm{ord}(Q(x)) = u$. If $u = s$, then there exist $x, y \in L$ such that $\mathrm{ord}(\langle x, y \rangle) = u$ and one of the three elements $Q(x)$, $Q(y)$ and $Q(x+y)$ must have ordinal u. Thus, we can find $x \in L$ such that $Q(x) = e\pi^u$ with $\mathrm{ord}(e) = 0$. By scaling L, we may assume that $e = 1$. Thus, $\pi^u \mathfrak{o}^2 \subseteq q(L)$. In case 3), we see easily that $s(L) = 0$ and $q(L) = \pi^u \mathfrak{o}^2$. In case 4), it is clear that v may be found to satisfy all the conditions. Since $\mathfrak{o}^2$-modules of finite type in k have ranks at most 2, e, u and v may always be found to satisfy the conditions stated.

The uniqueness of u follows from the equation $\pi^u \mathfrak{o} = q(L)\mathfrak{o}$. Let $\mathrm{rank}_{\mathfrak{o}^2} q(L) = 2$, then v is the minimal ordinal in the set $\{\mathrm{ord}(a) \mid a \in q(L)$ and $\mathrm{ord}(a) \equiv u + 1 \bmod 2\}$. Thus, v is also uniquely determined, provided that it satisfies the conditions stated. Q. E. D.

A subset $\{x_i \mid 1 \leqq i \leqq m\}$ of a free $\mathfrak{o}$-module M of finite rank is called pure if it generates a direct summand of rank m. Clearly, given a set $\{y_i \mid 1 \leqq i \leqq m\}$ of k-independent elements in kM, there is a pure subset $\{x_i \mid 1 \leqq i \leqq m\}$ in M such that for $p = 1, 2, \cdots, m$, $\sum_{1 \leqq i \leqq p} kx_i = \sum_{1 \leqq i \leqq p} ky_i$.

Let M be a quadratic lattice and $-\infty < i \leqq \infty$. Then M is called an i-modular lattice if $\{x\} \subseteq M$ is a pure subset implies that $\langle x, M \rangle = \pi^i \mathfrak{o}$. If M is i-modular, then, $i = \infty$ implies that $M = K(M)$; $i < \infty$ implies that $K(M) = 0$, $s(M) = \pi^i \mathfrak{o}$ and $\{x\} \subseteq M$ is a pure subset if and only if $\langle x, M \rangle = \pi^i \mathfrak{o}$. If $M = M_1 \oplus M_2$, then M is i-modular if and only if M_1 and M_2 are i-modular.

LEMMA 1.3. *Let L be a quadratic lattice and J be a sublattice of L. Then,*

1). *If J is i-modular, then J is an orthogonal summand of L if and only*

if, $J \subseteq L(i)$ and J is a direct summand of $L(\infty)$ when $i = \infty$.

2). *If L is i-modular with $i < \infty$, then J is an orthogonal summand of L if and only if J is i-modular.*

Proof. 1) follows easily as in [15; Proposition 1, p. 160]. 2) follows from 1) and the preceding remarks.

COROLLARY 1.4. *Let M be an i-modular lattice, $i < \infty$, then*

$$M \cong \oplus_j \begin{pmatrix} \pi^i & \\ a_j & c_j \end{pmatrix}$$

and conversely.

Proof. By induction on $\dim M$ and Lemma 1.3.

2. Orthogonal decompositions and change of bases.

Definition 2.1. Let L be a quadratic lattice. By an easy induction argument, using Lemma 1.3, we obtain an orthogonal decomposition $L = \bigoplus_{1 \leqq i \leqq m} L_i \oplus \bigoplus_{1 \leqq i \leqq p} K_i$, where $\dim L_i = 2$, $\dim K_i = 1$, $0 \leqq p \leqq 2$, L_i is $s(i)$-modular with $s(1) \leqq \cdots \leqq s(m) < \infty$, and $K(L) = \bigoplus_{1 \leqq i \leqq p} K_i$. By Corollary 1.4, we have $L = \bigoplus_{1 \leqq i \leqq n} M_i \oplus K(L)$, where M_i is $t(i)$-modular with $t(1) < \cdots < t(n) < \infty$. These decompositions will be called complete and canonical respectively.

Let $L_i = \mathfrak{o}x_i + \mathfrak{o}y_i$ and $K_i = \mathfrak{o}z_i$ in the complete decomposition. Then, the following transformations will be called elementary:

LT_0: a). $z'_i = z_{i+1}$; $z'_{i+1} = z_i$,

b). $z'_i = z_i + az_{i+1}$, $\operatorname{ord}(a) \geqq 0$,

c). $z'_i = az_i$, $\operatorname{ord}(a) = 0$,

d). $x'_m = x_m + az_j$, $\operatorname{ord}(a) \geqq 0$,

LT_1: a). $x'_i = y_i$, $y'_i = x_i$,

b). $x'_i = x_i + ay_i$, $\operatorname{ord}(a) \geqq 0$,

c). $x'_i = ax_i$, $\operatorname{ord}(a) = 0$,

LT_2: a). $x'_i = x_{i+1}$, $y'_i = y_{i+1}$; $x'_{i+1} = x_i$, $y'_{i+1} = y_i$, provided that $s(i) = s(i+1) = \operatorname{ord}(\langle x_i, y_i \rangle) = \operatorname{ord}(\langle x_{i+1}, y_{i+1} \rangle)$,

b). $x'_i = x_i + ax_{i+1}$, $y'_i = y_i$; $x'_{i+1} = x_{i+1}$, $y'_{i+1} = ab_{i+1}b_i^{-1}y_i + y_{i+1}$, where $\operatorname{ord}(a) \geqq 0$, $b_i = \langle x_i, y_i \rangle$ and $b_{i+1} = \langle x_{i+1}, y_{i+1} \rangle$.

It is to be understood that in each of the nine types of transformations, all the remaining basis elements remain unchanged. Transformations obtained by interchanging the roles of x_i and y_i will be called elementary of the same type.

It is easy to see that the new basis elements in the given order will again give a complete decomposition of L.

LEMMA 2.2. *Let L be a quadratic lattice with a complete decomposition as in Definition* 2.1. *Let $\{x\} \subseteq L$ be a pure subset such that $\langle x, L\rangle = s(L)$. Then there is a sequence of elementary transformations which will lead from the given basis to a new basis such that x is the first basis element.*

Proof. If $m = 0$ or 1, then the hypothesis that $\langle x, L\rangle = s(L)$ together with transformations of type LT_0 and LT_1 easily give the desired result. Thus, we proceed by induction on m and let $m > 1$. By using transformations of type LT_1, LT_2 a) and the hypothesis that $\langle x, L\rangle = s(L)$, it is easy to see that we may assume that $x = x_1 + \sum_{2 \leqq i \leqq m} \alpha_i x_i + \sum_{1 \leqq i \leqq p} \gamma_i z_i$, where $\alpha_i, \gamma_i \in \mathfrak{o}$. Using transformations of type LT_2 b), we may first assume that $\alpha_2 = 0$, then $\alpha_2 = 1$. Thus, by induction, we may assume that $x = x_1 + x_2$ and another application of LT_2 b) gives the desired result. Q. E. D.

THEOREM 2.3. *Let x_i, y_i, z_j and x'_i, y'_i, z'_j be the bases elements of two complete decompositions of a quadratic lattice L. Then, there is a sequence of elementary lattice transformations which lead from the first to the second set of basis elements.*

Proof. If $m = 0$, then our assertion reduces to the classical result on changing basis in a free $\mathfrak{o}$-module of finite type. If $m > 0$, then by Lemma 2.2, we may let $y'_1 = y_1$. It is clear that $\operatorname{ord}(\langle x_1, y_1\rangle) = \operatorname{ord}(\langle x'_1, y'_1\rangle) < \infty$. Hence, we may assume that x'_1 has the form given in the proof of Lemma 2.2. We now observe that the transformations carried out in Lemma 2.2 do not alter y_1. Hence, we may assume that $x'_1 = x_1$. Now by induction, we get the desired result. Q. E. D.

3. Hyperbolic lattices.

Definition 3.1. A quadratic space U is called hyperbolic if it is the orthogonal sum of hyperbolic planes, $\begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}_k$. If $i < \infty$, then a quadratic lattice L is called i-hyperbolic, when it is the orthogonal sum of i-hyperbolic planes, $\begin{pmatrix} \pi^i & \\ 0 & 0 \end{pmatrix}$. If M is any lattice, H is an i-hyperbolic plane, then $q_i(M \oplus H)$

$= q_i(M)$; in addition, if M is i-modular, then $Q(M \oplus H) = q(M \oplus H) = q(M) = q_i(M)$.

LEMMA 3.2. *A quadratic lattice L is i-hyperbolic if and only if, L is i-modular, kL is a hyperbolic space and $q(L) = \pi^i \mathfrak{o}$.*

Proof. The necessity is clear. Conversely, suppose that the three conditions hold. By Lemma 2.2, we may find $x, y \in L$ such that $M = \mathfrak{o}x + \mathfrak{o}y = \begin{pmatrix} b & \\ 0 & c \end{pmatrix}$, where $\operatorname{ord}(b) = i$, $\operatorname{ord}(c) \geqq i$. Replacing y by $\pi^i b^{-1}(cb^{-1}x + y) \in M$, we see that M is i-hyperbolic. By Lemma 1.3, $L = M \oplus M'$. By the cancellation theorem, [1; Satz 6, p. 157], kM' is hyperbolic. Thus, by Lemma 1.3 and Definition 3.1, we see that M' is i-modular and $q(M') = \pi^i \mathfrak{o}$, (provided that $M' \neq 0$.) Thus, by induction on $\dim L$, we get the desired result.

Q. E. D.

THEOREM 3.3. *Let $L = H_j \oplus L_j$, $j = 1, 2$, be two decompositions of a lattice L such that H_j is an i-hyperbolic plane for $j = 1, 2$. Then, $L_1 \cong L_2$.*

Proof. By scaling L, we may assume that $H_j = \mathfrak{o}x_j + \mathfrak{o}y_j = \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}$. Thus, $H_1 + H_2 \subseteq L(0)$ and $\langle H_1, H_2 \rangle \subseteq \mathfrak{o}$.

Case 1. $\langle H_1, H_2 \rangle = \mathfrak{o}$. Without loss of generality, we may assume that $\langle x_1, y_2 \rangle = 1$. Thus, $H' = \mathfrak{o}x_1 + \mathfrak{o}y_2 = \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \subseteq L(0)$. By Lemma 1.3, $L = H' \oplus L'$. Thus, by symmetry, we may assume that $x_1 = x_2$ and $y_1 \neq y_2$. Let z_j, $1 \leqq j \leqq m$, be a free $\mathfrak{o}$-basis for L_1. Then, $y_2 = ax_1 + by_1 + z$, where $z \in L_1$ and $1 = \langle x_2, y_2 \rangle = \langle x_1, y_1 \rangle = b$. Since $y_2 \in H_2 \subseteq L(0)$, we have $\langle y_2, z_j \rangle \in \mathfrak{o}$ and $w_j = \langle y_2, z_j \rangle x_1 + z_j \in L$ for $1 \leqq j \leqq m$. It is now trivial to see that $L = (\mathfrak{o}x_2 + \mathfrak{o}y_2) \oplus \sum_{1 \leqq j \leqq m} \mathfrak{o}w_j$. Thus, $L_2 = \sum_{1 \leqq j \leqq m} \mathfrak{o}w_j$. By computation, we see that the map $z_j \to w_j$ leads to an isometry between L_1 and L_2.

Case 2. $\langle H_1, H_2 \rangle \subseteq \pi\mathfrak{o}$. Let $\{z_1 + z_2\}$ be a pure subset of $H = H_1 + H_2$, where $z_j \in H_j$. Without loss of generality, let $\{z_1\}$ be a pure subset of H_1. From $\langle H_1, H_2 \rangle \subseteq \pi\mathfrak{o}$, $H \subseteq L(0)$, we see that $\mathfrak{o} = \langle z_1, H_1 \rangle = \langle z_1 + z_2, H_1 \rangle \subseteq \langle z_1 + z_2, H \rangle \subseteq \mathfrak{o}$. Thus, H is 0-modular. By Lemma 1.3, $L = H \oplus M$, $H = H_j \oplus M_j$; thus, we may assume that $L = H = H_1 + H_2 = H_j \oplus L_j$. Hence, $x_2 = w' + z'$ and $y_2 = w'' + z''$, where $w', w'' \in H_1$ and $z', z'' \in L_1$. Since $L = H_1 + H_2$, we see that $L_1 = \mathfrak{o}z' + \mathfrak{o}z''$. From $\langle H_1, H_2 \rangle \subseteq \pi\mathfrak{o}$, we see that $w', w'' \in \pi H_1$. From $Q(x_2) = Q(y_2) = 0$ we get $Q(z') = Q(w')$ and $Q(z'') = Q(w'')$. Thus, $Q(z'), Q(z'') \in q(\pi H_1) = \pi^2\mathfrak{o}$. We have shown that

$H = L$ is 0-modular, thus, L_1 is 0-modular by Lemma 1.3, hence, $\mathrm{ord}(\langle z', z''\rangle) = 0$. By Lemma 1.1, $\Delta(kL_1) = 0$, thus, kL_1 is a hyperbolic space, cf. [1; Zusatz 2, p. 153]. It is also clear that $q(L_1) = \mathfrak{o}$. Hence, by Lemma 3.2, L is 0-hyperbolic. By symmetry, L_2 is also 0-hyperbolic. Hence, $L_1 \cong L_2$.

Q. E. D.

4. Isometry of modular lattices.

LEMMA 4.1. *Let* $B_j = \mathfrak{o}x_j + \mathfrak{o}y_j = \begin{pmatrix} \pi^i & \\ a & c_j \end{pmatrix}$ *and* $q(B_j) \subseteq a\mathfrak{o}$, $j = 1, 2$. *Then* $B_1 \cong B_2$ *if and only if* $\Delta(kB_1) = \Delta(kB_2)$.

Proof. The necessity is obvious. Conversely, by scaling, we may assume that $i = 0$. Since $\mathfrak{o} \subseteq q(B_j) \subseteq a\mathfrak{o}$, we have $\mathrm{ord}(a) \leqq 0$ and $\mathrm{ord}(c_1 + c_2) \geqq \mathrm{ord}(a)$. Let $d \in k$ such that $a(c_1 + c_2) = \mathfrak{P}(da)$. Then $c_2 = c_1 + d^2a + d$ and $\mathrm{ord}(d) + \mathrm{ord}(1 + da) = \mathrm{ord}(c_1 + c_2) \geqq \mathrm{ord}(a)$. If $\mathrm{ord}(d) < 0$, then $\mathrm{ord}(d) + \mathrm{ord}(1 + da) < \mathrm{ord}(1 + da) = \mathrm{ord}(da) < \mathrm{ord}(a)$, a contradiction. Thus, $\mathrm{ord}(d) \geqq 0$, $dx_1 + y_1 \in B_1$ and we have $Q(dx_1 + y_1) = d^2a + d + c_1 = c_2$. Hence $B_1 \cong B_2$. Q. E. D.

COROLLARY 4.2. *Let* $B \cong \begin{pmatrix} 1 & \\ \pi^u & c \end{pmatrix}$ *be a quadratic lattice such that* $q(B) \subseteq \pi^u\mathfrak{o}$. *Let* $\pi^u c = \pi^{-1}b^2 + D + \mathfrak{P}(d)$ *as in Lemma* 1.1, 5). *Then* $B \cong \begin{pmatrix} 1 & \\ \pi^u & \pi^{-u}(\pi^{-1}b^2 + D) \end{pmatrix}$.

Proof. By Lemma 4.1 and Lemma 1.1, 5).

LEMMA 4.3. *Let* M *be an* i*-modular lattice,* $i < \infty$. *Then,* $\pi^{-i} \circ M \cong \begin{pmatrix} 1 & \\ 1 & D \end{pmatrix} \oplus H$, *where* $D = 0$ *or* λ *and* H *is* 0 *or* 0*-hyperbolic, if and only if,* $q(M) = \pi^i\mathfrak{o}$.

Proof. The necessity is obvious. By scaling, we may take $i = 0$. From Corollary 1.4 and that $q(M) = \mathfrak{o}$, we have $M \cong \bigoplus_{1 \leqq j \leqq m} \begin{pmatrix} 1 & \\ a_j & c_j \end{pmatrix}$, where $a_j, c_j \in \mathfrak{o}$, $1 \leqq j \leqq m$. If $a_jc_j \in \pi\mathfrak{o}$, then by Lemma 1.1 and Lemma 3.2, $\begin{pmatrix} 1 & \\ a_j & c_j \end{pmatrix} \cong \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}$. If $\mathrm{ord}(a_j) = \mathrm{ord}(c_j) = 0$, then an easy application of Lemma 4.1 and Lemma 1.1 shows that we may assume $c_j \neq 0 \in \Omega$. By symmetry, we may assume that $a_j \neq 0 \in \Omega$. Using a transformation of type $LT_1\,c)$, we may assume that $a_j = 1$. By Corollary 4.2, we may replace c_j by 0 or λ. As we have just seen $c_j = 0$ leads to a 0-hyperbolic plane. By using a transformation of type $LT_2\,b)$, $\begin{pmatrix} 1 & \\ 1 & \lambda \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. From this we get our assertion immediately. Q. E. D.

LEMMA 4.4. *Let M be an i-modular lattice, $i<\infty$. If $\dim M \geqq 6$, then M contains an i-hyperbolic plane.*

Proof. We may assume that $\dim M = 6$ and $i = 0$. Let $q(M) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$ be a standard representation of the norm group. Since $\operatorname{ord}(e) = 0$, by scaling, we may assume that $e = 1$ and $\pi^u = Q(x)$ for some $x \in M$, $u < v \leqq 1$, and $u + v \equiv 1 \bmod 2$. $\{x\}$ is a pure subset of M; hence, we can find $y \in M$ such that $\operatorname{ord}(\langle x, y\rangle) = 0$. From Lemma 1.3, $M = (\mathfrak{o}x + \mathfrak{o}y) \oplus N$. By the choice of x and using transformations of type LT_0 d) on $\mathfrak{o}x \oplus N$, we may assume that $\mathfrak{o}x \oplus N = (\pi^u) \oplus \begin{pmatrix} 1 & \\ a & b \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ c & d \end{pmatrix}$, where $a, b, c, d \in \pi^v \mathfrak{o}^2$, and $\operatorname{ord}(a) \leqq \operatorname{ord}(c) \leqq \operatorname{ord}(d)$. By transformations of type LT_0 d) on $(a) \oplus \begin{pmatrix} 1 & \\ c & d \end{pmatrix}$, we see that M contains a 0-hyperbolic plane.

Q. E. D.

LEMMA 4.5. *Let M be an i-module lattice, $i<\infty$, and $\dim M \geqq 4$. Let $q(M) = e(\pi^u \mathfrak{o}^2 + \pi^v \mathfrak{o}^2)$ be standard and $e\pi^u = Q(x)$. Then,*

1). $q(M) = Q(M)$.

2). $e^{-1}\pi^{-i} \circ M \cong \begin{pmatrix} 1 & \\ a_1 & c_1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ a_2 & c_2 \end{pmatrix} \oplus H$, *where H is 0-hyperbolic or zero and the following hold:*

Case 1. $v \leqq i$. $a_1 = \pi^{u-i}$, $c_1 = \pi^{-u+i} D_1 + \pi^{v+i} b^2$, $a_2 = \pi^{v-i}$, *and* $c_2 = \pi^{-v+i} D_2$, *where* $D_1, D_2 \in \{0, \lambda\}$ *and* $\operatorname{ord}(b) \geqq 0$.

Case 2. $v = i+1$ *and* $u < i$. a_1 *and* c_1 *as in Case* 1, $a_2 = c_2 = 0$.

Case 3. $v = i+1$ *and* $u = i$. $a_1 = 1$, $c_1 = D_1$, $a_2 = c_2 = 0$.

Proof. 1) follows from 2).

2). By scaling, we may assume that $i = 0$ and $e = 1$. By Lemma 4.4 and Definition 3.1, we may assume that $\dim M = 4$. $\{x\}$ is clearly a pure subset of M, thus, by Lemma 1.3 and Lemma 2.2, we have $M \cong \begin{pmatrix} 1 & \\ \pi^u & f \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ g & h \end{pmatrix}$. Using transformations of type LT_2 b), we may assume that $g, h \in \pi^v \mathfrak{o}^2$.

Case 3 now follows easily from Lemma 4.3.

Case 2. $v = 1$ and $u < 0$. By Lemma 3.2 (also the proof of Lemma 4.3), we may assume that $g = h = 0$. By Corollary 4.2, we may assume that $f = \pi^{-u}(\pi^{-1} b_1^2 + D)$, where $b_1 \in \Omega[\pi^{-1}]$ and $D = 0$ or λ. If $b_1 \neq 0$, then $\operatorname{ord}(f) + u \equiv 1 \bmod 2$, hence f has the desired form as c_1.

Case 1. Let $v=0$. By Lemma 4.3, we may assume that $g=1$ and $h=0$ or λ. Repeating Case 2, we get the desired result.

Let $v<0$ and $\mathrm{ord}(g)=v$. Using transformations of type LT_1 c), we may assume that $g=\pi^v$. As in Case 2, we may replace h by $\pi^{-v}(\pi^{-1}b_1{}^2+D)$. By choice, $\pi^{-v-1}b_1{}^2\in\pi^u\mathfrak{o}^2$. Thus, using a transformation of type LT_2 b), we may assume that $b_1=0$. One more application of Corollary 4.2 on $\begin{pmatrix}1\\ \pi^u & f\end{pmatrix}$ gives the desired form.

Finally, let $v<0$ and $v<\mathrm{ord}(g)\leqq\mathrm{ord}(h)$. Let $M'=\mathfrak{o}x+\mathfrak{o}y=\begin{pmatrix}1\\ \pi^u & f\end{pmatrix}\subseteq M$. Since $v<0$, we have $q(M')=q(M)$. Thus, for some $\alpha,\beta\in\mathfrak{o}$, $\mathrm{ord}(Q(\alpha x+\beta y))=v$. Clearly, $\{\alpha x+\beta y\}$ is a pure subset of M'. If $\mathrm{ord}(\beta)>0$, then $\mathrm{ord}(\alpha)=0$; hence,

$$\mathrm{ord}(Q(\alpha x+\beta y))=\mathrm{ord}(\alpha^2\pi^u+\alpha\beta+\beta^2 f)=u<v<0,$$

a contradiction. Thus, $\mathrm{ord}(\beta)=0$. Replacing y by $\beta^{-1}(\alpha x+\beta y)$, we may assume that $\mathrm{ord}(f)=v$. Using a transformation of type LT_2 b), we have $M\cong\begin{pmatrix}1\\ \pi^u+g & f\end{pmatrix}\oplus\begin{pmatrix}1\\ g & h+f\end{pmatrix}$, where $h+f=\pi^v f_1{}^2$, $\mathrm{ord}(f_1)=0$. Using transformations of type LT_1, $M\cong\begin{pmatrix}1\\ \pi^u+g & f\end{pmatrix}\oplus\begin{pmatrix}1\\ \pi^v & h'\end{pmatrix}$, where $g,h'\in\pi^v\mathfrak{o}^2$. Using a transformation of type LT_2 b), we have $M\cong\begin{pmatrix}1\\ \pi^u & f'\end{pmatrix}\oplus\begin{pmatrix}1\\ \pi^v & h''\end{pmatrix}$. Using one more transformation of type LT_2 b), we may assume that $h''\in\pi^v\mathfrak{o}^2$. Repeating the argument of the preceding paragraph, we get the desired result.

Q. E. D.

THEOREM 4.6. *Let M_1 and M_2 be i-modular lattices. Then, $M_1\cong M_2$ if and only if $kM_1\cong kM_2$ and $q(M_1)=q(M_2)$.*

Proof. The necessity is obvious. If $i=\infty$, then $q(M_j)=Q(M_j)$, $j=1,2$. It is easy to see that either $M_j\cong(e\pi^u)$ or $M_j\cong(e\pi^u)\oplus(e\pi^v)$, where $q(M_j)=e(\pi^u\mathfrak{o}^2+\pi^v\mathfrak{o}^2)$ is a standard representation.

Thus, let $i<\infty$, $kM_1\cong kM_2$, $q(M_1)=q(M_2)$ and M_1, M_2 be i-modular. By Definition 3.1 and the cancellation theorem, cf. [1; Zusatz 2, p. 153], these conditions are "invariant" under addition or deletion of i-hyperbolic planes. Thus, by Theorem 3.3 and Lemma 4.4, we may assume that $\dim M_1=\dim M_2=4$. By scaling, we may assume that $i=0$ and $q(M_j)=\pi^u\mathfrak{o}^2+\pi^v\mathfrak{o}^2$, $u<v\leqq 1$, $u+v\equiv 1 \bmod 2$, $j=1,2$. If $v=1$, then by Lemma 4.5, 2) Case 3 and a simple computation of the Arf invariant of kM_1 and kM_2, we see that $M_1\cong M_2$. Thus, we may assume that $v\leqq 0$. By Corollary 4.2 and Lemma 4.5, 2),

$$M_j \cong \begin{pmatrix} 1 & \\ \pi^u & \pi^{-u}(\pi^{-1}b_j^2 + D'_j) \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \pi^v & \pi^{-v}D_j \end{pmatrix},$$

where $b_j \in \Omega[\pi^{-1}]$, $D'_j, D_j \in \{0, \lambda\}$, $j = 1, 2$.

If u is even, then v is odd. Thus, $[C(kM_j)] = [\pi^v, \pi^{-u}D_j] = [\pi^{-1}, \pi D_j]$ for $j = 1, 2$. Hence, from $kM_1 \cong kM_2$, we get $D_1 = D_2$. From Lemma 4.1 and that $\Delta(kM_1) = \Delta(kM_2)$, we see that $M_1 \cong M_2$.

If u is odd, then v is even. Thus, $[C(kM_j)] = [\pi^u, \pi^{-u}D'_j] = [\pi^{-1}, \pi D'_j]$ for $j = 1, 2$. As above, $D'_1 = D'_2$. From $kM_1 \cong kM_2$, we get $0 = \Delta(kM_1) + \Delta(kM_2) = \pi^{-1}(b_1 + b_2)^2 + (D_1 + D_2) + \mathfrak{P}k$. By Lemma 1.1, $b_1 = b_2$ and $D_1 = D_2$. Thus, $M_1 \cong M_2$. Q. E. D.

5. Isometry of lattices.

Definition 5.1. Let $L = \underset{0 \leqq i \leqq m}{\oplus} L_i \oplus K(L)$ be a canonical decomposition, where L_j is $s(j)$-modular, $s(0) < \cdots < s(m) < \infty$. It is easy to see that for i such that $s(n) \leqq i < s(n+1)$, we have

$$L(i) = \underset{0 \leqq j \leqq n}{\oplus} \pi^{i-s(j)}L_j \oplus \underset{n+1 \leqq j \leqq m}{\oplus} L_j \oplus K(L).$$

Let $q_i(L) = e_i(\pi^{u(i)}\mathfrak{o}^2 + \pi^{v(i)}\mathfrak{o}^2)$ be standard representations of the i-th norm group for $i \leqq \infty$, i.e. they are standard representations of $q(L(i) \oplus H_i)$ or of $q(K(L))$, where H_i is an i-hyperbolic plane, when $i < \infty$, we do not assert that $e_i\pi^{u(i)} \in Q(L(i))$. Clearly, we have $q_i(L) \supseteq q_{i+1}(L) \supseteq \cdots \supseteq q_\infty(L)$ and $\pi^2 q_i(L) \supseteq q_{i+1}(L)$. From these we deduce easily the following possibilities:

1). $u(j+1) = u(j)$. Then, $v(j+1) = v(j)$ or $v(j)+2$ and $= v(j)$ when $v(j) = j+1$, and $q_j(L) = e_{j+1}(\pi^{u(j)}\mathfrak{o}^2 + \pi^{v(j)}\mathfrak{o}^2)$.

2). $u(j+1) = u(j)+1$. Then, $v(j+1) = v(j)+1 = u(j+1)+1$ and $q_j(L) = \pi^{u(j)}\mathfrak{o}$, $q_{j+1}(L) = \pi^{u(j)+1}\mathfrak{o} = \pi^{u(j+1)}\mathfrak{o}$.

3). $u(j+1) = u(j)+2$. Then, $q_{j+1}(L) = e_j(\pi^{u(j+1)}\mathfrak{o}^2 + \pi^{v(j)}\mathfrak{o}^2)$.

The given canonical decomposition is said to be saturated in the $s(j)$-th component, if $q_{s(j)}(L) = Q(L_j) = q(L(s(j)))$. It is said to be saturated, if it is saturated in each $s(i)$-th component, $0 \leqq i \leqq m$.

If $L = K(L)$, then let $F(L) = T(L) = \infty$ and $D(L) = (\infty)$. If $L \neq K(L)$, then let $F(L) = s(0)$, $T(L) = s(m) + n$, where n is the largest finite integer among 0, $u(\infty) - u(s(m))$ and $u(\infty) - u(s(m)) + v(\infty) - v(s(m))$, and $D(L) = (r_{F(L)}, \cdots, r_{T(L)})$, where $r_i = 0$ unless $i = s(j)$, then $r_i = \dim L_j$. $(F(L), T(L))$ and $D(L)$ are called the type and the type dimension of L respectively. L is called a normal lattice when $r_i \geqq 8$ for

$F(L) \leqq i \leqq T(L)$. Thus, either $L = K(L)$ or when $L \neq K(L)$, then $u(\infty) = u(T)$, $v(\infty) = v(T)$ or ∞, and $s(j) = F + j$, $0 \leqq j \leqq T - F$. Clearly, $F(L)$, $T(L)$ and $D(L)$ are invariants of L.

LEMMA 5.2. *Let* $L = \bigoplus_{0 \leqq i \leqq m} L_i \oplus K(L)$ *be a canonical decomposition of the lattice* L *of type* (F, T). *Let* $H = 0$ *when* $F = T = \infty$ *and* $H = \bigoplus_{F \leqq j \leqq T} H_j$, *where* H_j *is* j*-hyperbolic of dimension* 8 *otherwise. Then,*

1). $q_i(L \oplus H) = q_i(L)$ *for* $i = F, F+1, \cdots, T, \infty$.

2). $L \oplus H$ *is a normal lattice of type* (F, T).

Proof. If $L = K(L)$, then there is nothing to prove. Thus, we assume that $(F, T) \neq (\infty, \infty)$.

1). follows from the definition of $q_i(L)$.

2). If $K(L) = 0$, then the normality follows easily from 1). Thus, let $K(L) \neq 0$ and n be chosen such that $T(L) = s(m) + n = T$.

From Definition 5.1 and 1), $q_T(L \oplus H) = q(\pi^n L(s(m))) + q_\infty(L) + \pi^T \mathfrak{o}$. Since $L_m \neq 0$ is $s(m)$-modular, let

$$q(L(s(m))) = q_{s(m)}(L) = e_{s(m)}(\pi^{u(s(m))}\mathfrak{o}^2 + \pi^{v(s(m))}\mathfrak{o}^2)$$

be a standard representation. By the choice of n, Definition 5.1 and that $K(L) \neq 0$, we see that $T = s(m) + n \geqq u(\infty) + [s(m) - u(s(m))] \geqq u(\infty)$ and $2n + u(s(m)) \geqq u(\infty) + [u(\infty) - u(s(m))] \geqq u(\infty)$. Thus, $q_T(L \oplus H)\mathfrak{o} = q_\infty(L)\mathfrak{o}$, or $u(T) = u(\infty)$. If $v(\infty) = \infty$, then by 1) and definition, $L \oplus H$ is normal. Let $v(\infty) < \infty$. If $u(\infty) \geqq v(s(m))$, then $n \geqq v(\infty) - u(s(m))$, thus, we get $T \geqq v(\infty)$ and $2n + u(s(m)) \geqq v(\infty)$. From $\pi^{v(\infty)}\mathfrak{o} \subseteq q_\infty(L)$ we get $q_T(L \oplus H) = q_\infty(L)$, thus $L \oplus H$ is normal by 1). Finally, if $u(\infty) < v(s(m))$, then $u(s(m)) \equiv u(\infty)$ and $v(s(m)) \equiv v(\infty)$ mod 2. The argument used in case $v(\infty) = \infty$ shows that we may assume $u(s(m)) = u(\infty)$. Hence, we may assume that $e_{s(m)} = e_\infty$. It is now trivial to see that $q_T(L \oplus H) = q_\infty(L)$, thus, $L \oplus H$ is normal. Q. E. D.

LEMMA 5.3. *A normal lattice* L *admits a saturated canonical decomposition.*

Proof. Let $L = \bigoplus_{0 \leqq i \leqq m} L_i \oplus K(L)$ be a canonical decomposition. If $L = K(L)$, then the decomposition is already saturated. Thus, we may assume that $L \neq K(L)$. By scaling, we may assume that L has type $(0, m)$. If $m = 0$ and $K(L) = 0$, then Lemma 4.5 furnishes the desired result. If

$m=0$ and $K(L)\neq 0$, then by Lemma 4.4 and Lemma 4.5, we may assume that $L=L'_0\oplus H\oplus K(L)$, where H is 0-hyperbolic of dimension 4 and $Q(L'_0)=q(L_0)$. Let $q(K(L))=e(\pi^u\mathfrak{o}^2+\pi^v\mathfrak{o}^2)$ be a standard representation. Then by using transformations of type LT_0 on $H\oplus K(L)$, we have $H\oplus K(L)\cong H'\oplus K(L)$, where $H'\cong\begin{pmatrix}1\\ e\pi^u\ 0\end{pmatrix}\oplus\begin{pmatrix}1\\ e\pi^v\ 0\end{pmatrix}$, since $Q(K(L))=q(K(L))$. Thus, $L=L'\oplus K(L)$, where $L'\cong L'_0\oplus H'$. It is clear that $q(L')=q(L)=q_0(L)$ and $L\cong L'\oplus K(L)$ is a saturated canonical decomposition. Hence, we may assume that $m>0$.

By Lemma 4.4 and Lemma 4.5, let $L_i=L'_i\oplus H_i$, where H_i is i-hyperbolic of dimension 4 and $q(L_i)=Q(L'_i)$, $i=0,1$. By Theorem 4.6, $L_0\cong L'_0\oplus H'_0$, where $H'_0\cong\begin{pmatrix}1\\ a\ 0\end{pmatrix}\oplus\begin{pmatrix}1\\ c\ 0\end{pmatrix}$, $a,c\in q(L_0)$. By using transformations of type LT_2, we see that $H'_0\oplus H_1\cong H'_0\oplus H'_1$, where

$$H'_1\cong\begin{pmatrix}1\\ \pi^2a\ 0\end{pmatrix}\oplus\begin{pmatrix}1\\ \pi^2c\ 0\end{pmatrix}.$$

Thus, from Definition 5.1, we see that we may assume that $L=L_0\oplus L'$ with $q_i(L')=q_i(L)$, $i=1,\cdots,m,\infty$. Hence, L' is a normal lattice of type $(1,m)$. By induction, we may then assume that the chosen canonical decomposition is already saturated in the i-th component, $1\leqq i\leqq m$. Now, by the same argument, $L_1\cong L'_1\oplus H''_1$, where $H''_1\cong\begin{pmatrix}\pi\\ b\ 0\end{pmatrix}\oplus\begin{pmatrix}\pi\\ d\ 0\end{pmatrix}$ and $H_0\oplus H''_1\cong H''_0\oplus H''_1$, where $H''_0\cong\begin{pmatrix}1\\ b\ 0\end{pmatrix}\oplus\begin{pmatrix}1\\ d\ 0\end{pmatrix}$ and $b,d\in q_1(L)$ are arbitrary. Thus, we can saturate the 0-th component without destroying the saturation of the remaining components. Hence our assertion holds.

Q. E. D.

LEMMA 5.4. *Let* $L=\bigoplus_{0\leqq i\leqq m}L_i\oplus K(L)=\bigoplus_{0\leqq i\leqq m}M_i\oplus K(L)$ *be two canonical decompositions of the normal lattice* L *of type* (F,T). *Let* $q_i(L)=e_i(\pi^{u(i)}\mathfrak{o}^2+\pi^{v(i)}\mathfrak{o}^2)$, $i=F,F+1,\cdots,T,\infty$, *be standard representations, cf. Definition* 5.1. *Then,*

1). $\operatorname{Ord}\Big(\sum_{0\leqq j\leqq i-1}(\Delta(kL_j)+\Delta(kM_j))\Big)$
$$\geqq u(F+i)+v(F+i)-(2F+2i+1),$$

2). *If* $v(F+i)=F+i+1$, *then*
$$\bigoplus_{0\leqq j\leqq i-1}kL_j\oplus(e_{F+i}\pi^{u(F+i)})_k\cong\bigoplus_{0\leqq j\leqq i-1}kM_j\oplus(e_{F+i}\pi^{u(F+i)})_k,$$

where $1\leqq i\leqq T-F=m$. ($\infty-\infty$ *is taken to mean* 0.)

Proof. Clearly, the assertions are "invariant" under scalings of L. By Theorem 2.3, we may assume that the given decompositions are related by an elementary lattice transformation. By inspection, it is clear that we need only consider transformations of type $LT_2\,b)$. By scaling L and using transformations of type LT_1, we may assume that the given decompositions of L differ as follows:

$$\begin{pmatrix} 1 & \\ a & b \end{pmatrix} \oplus \begin{pmatrix} \pi & \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & \\ a+cg^2 & b \end{pmatrix} \oplus \begin{pmatrix} \pi & \\ c & g^2\pi^2 b + d \end{pmatrix}, \quad \mathrm{ord}(g) \geqq 0;$$

it is to be understood that the given canonical decompositions are, in reality, complete decompositions, and the lattices above are orthogonal summands of the respective complete decompositions.

1). It suffices to show that $\mathrm{Ord}(bcg^2 + \mathfrak{P}k) \geqq u(1) + v(1) - 3$. From Definition 5.1, we see that $bcg^2 \in q_0(L)q_1(L) = \pi^{u(0)+u(1)}\mathfrak{o}^2 + \pi^{u(0)+v(1)}\mathfrak{o}^2 + \pi^{v(0)+u(1)}\mathfrak{o}^2 + \pi^{v(0)+v(1)}\mathfrak{o}^2$. If $u(1) = u(0) + 1$, then by 2) of Definition 5.1, $\mathrm{ord}(bcg^2) \geqq u(0) + u(1) \geqq u(1) + v(1) - 3$. Thus, we may assume that $u(1) - u(0), v(1) - v(0) \in \{0, 2\}$. Hence, $bcg^2 = \pi^{u(0)+u(1)}f^2 + h$, where $\mathrm{ord}(f) \geqq 0$ and $\mathrm{ord}(h) \geqq u(1) + v(1) - 3$. Let $n = \frac{1}{2}(u(0) + u(1)) \geqq u(1) - 1$, $b' = bcg^2 + \mathfrak{P}(\pi^n f)$. Since $v(1) \leqq 2$, we have $n \geqq u(1) + v(1) - 3$ and $\mathrm{Ord}(bcg^2 + \mathfrak{P}k) \geqq \mathrm{ord}(b') = \mathrm{ord}(\pi^n f + h) \geqq u(1) + v(1) - 3$.

2). By scaling, we may assume that $e_1 = 1$. By the cancellation theorem [1; Folgerung, p. 160], it suffices to show that,

$$\begin{pmatrix} 1 & \\ a & b \end{pmatrix}_k \oplus (\pi^{u(1)})_k \cong \begin{pmatrix} 1 & \\ a+cg^2 & b \end{pmatrix}_k \oplus (\pi^{u(1)})_k$$

when $v(1) = 2$. From [1; Satz 8, p. 161], it is enough to show that $[\pi^{-u(1)}cg^2, \pi^{u(1)}b] = 1$. From Definition 5.1 and that $v(1) = 2$, we see that $v(0) = 0$ or 1 (note that $v(0) \leqq 1$ by definition).

If $v(0) = 1$, then $u(1) = 1$ and $b \in \mathfrak{o}$, $cg^2 \in \pi\mathfrak{o}$. Thus, $[\pi^{-1}cg^2, \pi b] = [C(V)]$, where $V = \begin{pmatrix} 1 & \\ \pi^{-1}cg^2 & \pi b \end{pmatrix}_k$. By Lemma 1.1, $\Delta(V) = 0$, thus V is a hyperbolic plane [1; Zusatz 2, p. 153], hence $[C(V)] = 1$ as observed in Section 1.

If $v(0) = 0$, then $u(1) = u(0)$ or $u(0) + 2$. If $u(1) = u(0)$, then we may take $e_0 = 1$, by using Definition 5.1, 1). If $u(1) = u(0) + 2$, then by Definition 5.1, we have $\pi^2 q_0(L) = e_0 q_1(L) \subseteq q_1(L)$; now, from $\mathrm{ord}(e_0) = 0$, we have $q_1(L) = e_0^2 q_1(L) \subseteq e_0 q_1(L)$; hence, $q_0(L) = \pi^{-2}q_1(L)$ and we may take $e_0 = 1$. Thus, in all cases, $b \in \pi^{u(0)}\mathfrak{o}^2 + \pi^{v(0)}\mathfrak{o}^2$ and $cg^2 \in \pi^{u(1)}\mathfrak{o}^2 + \pi^{v(1)}\mathfrak{o}^2$, where $v(1) = 2$ and $v(0) = 0$. By choice, $u(0)$ and $u(1)$ must both be odd.

Thus, $[\pi^{-u(1)}cg^2, \pi^{u(1)}b] = [\pi^{-u(1)+v(1)}c_2{}^2, \pi^{u(1)+v(0)}b_2{}^2]$, where $b_2, c_2 \in \mathfrak{o}$. Using the same argument of the preceding paragraph, we get the desired result.

Q. E. D.

THEOREM 5.5. *Let* $L = \bigoplus_{0 \leqq i \leqq m} L_i \oplus K(L)$ *and* $M = \bigoplus_{0 \leqq i \leqq n} M_i \oplus K(M)$ *be canonical decompositions. Then* $L \cong M$ *if and only if,*

1). $kL \cong kM$.

2). L *and* M *have the same type* (F, T) *and* $D(L) = D(M)$. *In particular,* $m = n \leqq T - F$ *when* $(F, T) \neq (\infty, \infty)$.

3). $q_i(L) = q_i(M) = e_i(\pi^{u(i)}\mathfrak{o}^2 + \pi^{v(i)}\mathfrak{o}^2)$, $i = F, \cdots, T, \infty$, *where the representations are standard, cf. Definition* 5.1.

4). $$\mathrm{Ord}\Big(\sum_{0 \leqq j \leqq p(i)} (\Delta(kL_j) + \Delta(kM_j))\Big) \geqq u(F+i) + v(F+i) - (2F + 2i + 1),$$

where $p(i)$ *satisfies* $s(p(i)) < F + i \leqq s(p(i)+1)$, L_j *is* $s(j)$*-modular and* $1 \leqq i \leqq T - F$.

5). *If* $v(F+i) = F + i + 1$, *then*

$$\bigoplus_{0 \leqq j \leqq p(i)} kL_j \oplus (e_{F+i}\pi^{u(F+i)})_k \cong \bigoplus_{0 \leqq j \leqq p(i)} kM_j \oplus (e_{F+i}\pi^{u(F+i)})_k,$$

where $p(i)$ *is as in* 4), *and* $1 \leqq i \leqq T - F$.

Proof. Using Theorem 3.3, we see that our assertion is invariant under simultaneous addition of i-hyperbolic planes to L and M, provided that $F \leqq i \leqq T$. Thus, by Lemma 5.2, we may assume that L and M are both normal lattices.

Necessity. By Lemma 5.4.

Sufficiency. A_1). By the necessity above and Lemma 5.3, we may assume that the given decompositions are saturated.

A_2). By 1) and 3), we may assume that

$$K(L) \cong K(M) \cong (e_\infty \pi^{u(\infty)}) \oplus (e_\infty \pi^{v(\infty)}),$$

where (0) is understood to be the zero lattice. In particular, we may assume that $(F, T) \neq (\infty, \infty)$.

A_3). By Lemma 4.4, we may assume that $L_i = L'_i \oplus H_i$, where $\dim L'_i = 4$ and H_i is i-hyperbolic. Using Lemma 4.5, we may assume that L'_i is

presented in the form stated in 2) of Lemma 4.5. By A_1), the quantities e, u and v of Lemma 4.5 correspond to e_i, $u(i)$ and $v(i)$ respectively. From Theorem 4.6, we see that $L_i \cong L'_i \oplus H'_i \oplus H''_i$, where

$$H'_i \cong \begin{pmatrix} & \pi^i \\ e_i\pi^{u(i)} & 0 \end{pmatrix}\begin{pmatrix} & \pi^i \\ \oplus e_i\pi^{v(i)} & 0 \end{pmatrix}$$

and H''_i is either zero or i-hyperbolic. Similarly for M_i.

A_4). We may always consider $a \circ L$ and $a \circ M$ in place of L and M, since the given conditions imply a similar set of conditions for the scaled lattices and the latter will also be normal and satisfy A_1), A_2) and A_3). In particular, we may assume that $(F, T) = (0, m)$ and proceed by induction on $T - F = m$.

A_5). $(F, T) = (0, 0)$. Then, the following cases occur:

Case 1. $K(L) = K(M) = 0$. By Theorem 4.6, $L \cong M$.

Case 2. $K(L) \cong K(M) \cong (e_\infty \pi^{u(\infty)})$, $u(\infty) < \infty$.

By normality, $u(\infty) = u(0)$. By scaling, we may assume that $e_\infty = e_0 = 1$. From $u(0) \leqq 0$, Lemma 4.5 and using transformations of type LT_0 d) on $L'_0 \oplus K(L)$, we have,

$$L'_0 \oplus K(L) \cong \begin{pmatrix} 1 & \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \pi^{v(0)} & \pi^{-v(0)}D \end{pmatrix} \oplus K(L), \text{ if } v(0) \leqq 0,$$

$$L'_0 \oplus K(L) \cong \begin{pmatrix} 1 & \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus K(L), \text{ if } u(0) < 0 \text{ and } v(0) = 1,$$

$$L'_0 \oplus K(L) \cong \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus K(L), \text{ if } u(0) = 0 \text{ and } v(0) = 1,$$

where $c \in \pi^{v(0)}\mathfrak{o}^2$ and $D = 0$ or λ. Applying Theorem 4.6 to $\begin{pmatrix} 1 & \\ 0 & c \end{pmatrix} \oplus H'_0$, we may assume that $c = 0$. Since $u(0) + v(0) \equiv 1 \bmod 2$ and $u(\infty) = u(0)$, we see that $[\pi^{-1}, \pi D] = [C(\pi^{u(\infty)} \circ kL)/\mathfrak{R}]$, where $\mathfrak{R}$ is the radical of the Clifford algebra $C(\pi^{u(\infty)} \circ kL)$. By condition 1) and interchanging L and M, we see that $L \cong M$.

Case 3. $K(L) \cong K(M) \cong (e_\infty \pi^{u(\infty)}) \oplus (e_\infty \pi^{v(\infty)})$, $v(\infty) < \infty$. By normality, $u(\infty) = u(0)$ and $v(\infty) = v(0)$. Thus, we see easily that $q_0(L) = q_\infty(L) = q_\infty(M) = q_0(M)$. By using transformations of type LT_0 d), we may assume that L'_0 and M'_0 are 0-hyperbolic. Thus, $L \cong M$.

A_6). $(F, T) = (0, m)$, $m > 0$. We assert that $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ contain isometric 0-modular sublattices of dimension 4, say, $L''_0 \cong M''_0$. We have the following cases to consider:

Case 1. $v(0) = v(1)$. Since $v(0) \leqq 1$, condition 5) is vacuous on the 0-modular components. Thus, by Definition 5.1 and scaling by e_0 or e_1, we may assume that $e_0 = e_1 = 1$. Now $L'_0 \oplus H'_1$ contains the sublattice $N = L'_0 \oplus (\pi^{u(1)}) \oplus (\pi^{v(1)})$. By Lemma 4.5 and using transformations of type LT_0 d) on N, we see easily that N contains $\begin{pmatrix} 1 & \\ \pi^{u(0)} & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}$. Interchanging L and M, we get A_6).

Case 2. $v(0) + 1 = v(1)$. By Definition 5.1, $v(0) = u(0) + 1 = u(1)$ and scaling by e_1 allows us to assume that $e_0 = e_1 = 1$. If $v(0) \leqq 0$, then the argument of Case 1 may be used to get A_6). Thus, we may assume that $v(0) = 1$, hence $v(1) = 2$. By Lemma 4.5, we see that $L'_0 \cong \begin{pmatrix} 1 & \\ 1 & D \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}$ and $M'_0 \cong \begin{pmatrix} 1 & \\ 1 & D' \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}$, where $D, D' \in \{0, \lambda\}$. Condition 5), [1; Satz 8, p. 161] and the remarks preceding Lemma 1.1 show that $D = D'$. Thus A_6) holds.

Case 3. $v(0) + 2 = v(1)$. Since $v(1) \leqq 2$, we have $v(0) \leqq 0$. By Definition 5.1, $u(0) = u(1)$ or $u(1) - 2$. By scaling, we may assume that $e_1 = 1$.

Let $u(0) = u(1)$. If $v(1) < 2$, then $v(0) < 0$. Thus, an argument similar to that of Case 1 shows that $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ contain the sublattice $\begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \pi^{v(0)} & 0 \end{pmatrix}$. Hence, we may assume that $v(1) = 2$ and $v(0) = 0$. By Lemma 4.5, we see that $L'_0 \oplus (\pi^{u(1)}) \cong \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 1 & D \end{pmatrix} \oplus (\pi^{u(1)})$, where $u(1)$ is odd. By condition 5) and the second argument of Case 2, we see that $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ both contain the sublattice $\begin{pmatrix} 1 & \\ 1 & D \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}$. Thus, A_6) holds in either case.

Let $u(0) = u(1) - 2$. By Definition 5.1, we then have

$$e_0 q_1(L) = \pi^2 q_0(L) \subseteq q_1(L) = e_0^2 q_1(L) \subseteq e_0 q_1(L).$$

Thus, $q_0(L) = \pi^{-2} q_1(L)$ and we may assume that $e_0 = 1$ as well. If $v(1) < 2$, then $u(0) < v(0) < 0$, $-u(0) \geqq u(1)$ and $-v(0) \geqq v(1)$. Repeating the argument of Case 1, we see that $L'_0 \oplus H'_1$ contains the sublattice

$$\begin{pmatrix} 1 & \\ \pi^{u(0)} & \pi^{v(0)} b \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \pi^{v(0)} & 0 \end{pmatrix}, \text{ where } b \in \Omega.$$

Similarly, $M'_0 \oplus H'_1$ contains the sublattice

$$\begin{pmatrix} 1 & \\ \pi^{u(0)} & \pi^{v(0)}b' \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ \pi^{v(0)} & 0 \end{pmatrix}, \text{ where } b' \in \Omega.$$

From $0 > u(0) + v(0) \equiv 1 \bmod 2$, condition 4) and Lemma 1.1, we see that $b = b'$. Finally, if $v(1) = 2$, then $u(0) < v(0) = 0$ and $-u(0) \geqq u(1)$. Repeating the argument of Case 1, we see that $L'_0 \oplus H'_1$ contains the sublattice $\begin{pmatrix} 1 & \\ \pi^{u(0)} & b \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 1 & D \end{pmatrix}$ and that $M'_0 \oplus H'_1$ contains the sublattice $\begin{pmatrix} 1 & \\ \pi^{u(0)} & b' \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 1 & D' \end{pmatrix}$, where $b, b' \in \Omega$; $D, D' \in \{0, \lambda\}$; and $u(0) < 0$ is odd. By condition 5) and the second argument of Case 2, we get $D = D'$. By condition 4) and the last argument of the preceding paragraph, we see that $b = b'$. Thus A_6) holds.

From A_6), applying Lemma 1.3, we see that L''_0 and M''_0 are orthogonal summands of $L'_0 \oplus H'_1$ and $M'_0 \oplus H'_1$ respectively. Thus, we may "replace" L'_0 and M'_0 by L''_0 and M''_0 respectively. Hence, we have obtained decompositions of L and M satisfying A_1), A_2) as well as $L_0 \cong M_0$. Let L' and M' be the corresponding orthogonal complement of L_0 and M_0 in L and M. It is clear that L' and M' are normal lattices of type $(1, m)$ and satisfy a similar set of conditions as stated in our assertion. Hence, by induction, $L' \cong M'$. Thus, $L \cong M$. Q. E. D.

PRINCETON UNIVERSITY.

REFERENCES.

[1] C. Arf, "Untersuchungen über quadratische Formen in Körpern der Charakteristik 2," *Journal für die reine und angewandte Mathematik*, vol. 183 (1940), pp. 148-167.

[2] ———, "Über arithmetische Aequivalenz quadratische Formen in Potenzreihenkörpern über einem vollkommen Körpern der Charakterisk 2," *Revue de la Faculté des Sciences de l'Université d'Istanbul*, vol. 8 (1943), pp. 297-327.

[3] E. Artin, *Algebraic Numbers and Algebraic Functions*, New York University, 1951.

[4] ———, *Geometric Algebra*, New York, 1957.

[5] N. Bourbaki, "Algèbre," Chapitre 8, 9, *Actualités Scientifiques et Industrielles*, Nos. 1261, 1272, Paris, 1958-1959.

[6] P. Cartier, "On Arf and Dickson Invariants," (unpublished).

[7] C. Chevalley, *Introduction to the Theory of Algebraic Functions of One Variable*, New York, 1951.

[8] J. Dieudonné, *Sur les groupes classiques*, Paris, 1948.

[9] M. Eichler, *Quadratische Formen und orthogonalen Gruppen*, Berlin, 1952.

[10] H. Hasse, "Über die Äquivalenz quadratische Formen im Körper der rationalen Zahlen," *Journal für die reine und angewandte Mathematik*, vol. 152 (1923), pp. 205-224.

[11] ———, "Äquivalenz quadratische Formen in einem beliebigen algebraischen Zahlkörper," *ibid.*, vol. 153 (1924), pp. 158-162.

[12] W. Klingenberg and E. Witt, "Über die Arfsche Invariante quadratische Formen mod 2," *ibid.*, vol. 193 (1954), pp. 121-122.

[13] H. Minkowski, *Gesammelte Abhandlungen I*, Leipzig, 1911.

[14] O. T. O'Meara, "Quadratic forms over local fields," *American Journal of Mathematics*, vol. 77 (1955), pp. 87-116.

[15] ———, "Integral equivalence of quadratic forms in ramified local fields," *ibid.*, vol. 79 (1957), pp. 157-186.

[16] E. Witt, "Theorie der quadratische Formen in beliebigen Körpern," *Journal für die reine und angewandte Mathematik*, vol. 176 (1937), pp. 31-44.

[17] ———, "Zyklische Körper und Algebren der Charakteristik p vom Grad p^n," *ibid.*, vol. 176 (1937), pp. 126-140.

[18] ———, "Über eine Invariante quadratische Formen mod 2," *ibid.*, vol. 193 (1954), pp. 119-120.

A GENERALIZED THEOREM OF KRULL-SEIDENBERG ON PARAMETRIZED ALGEBRAS OF FINITE TYPE.*

By Heisuke Hironaka.

---

1. Introduction. Let B be a *Noetherian ring* and A a *commutative B-algebra* such that $1_B 1_A = 1_A$, where 1_B and 1_A denote the unities of B and A respectively. We have a canonical homomorphism $i: B \to A$ such that $i(b)a = ba$ for $b \in B$ and $a \in A$. We shall say that $i: B \to A$ is *of finite type*, meaning that A is a commutative B-algebra of finite type, i. e., a homomorphic image of a polynomial ring over B. In general, if A' is a commutative ring with unity, $Spec(A')$ denotes the set of all prime ideal in A' which is furnished with *Zariski topology*; a subset X of $Spec(A')$ is, by defintion, open (respectively closed) if there exists an ideal $\mathfrak{A}$ in A' such that X is the set of those prime ideals in A' which do not (respectively do) contain $\mathfrak{A}$, where $\mathfrak{A}$ may be A' itself. The homomorphism $i: B \to A$ gives rise to a continuous mapping $f: Spec(A) \to Spec(B)$ defined by $f(\mathfrak{P}) = i^{-1}(\mathfrak{P})$ (= the prime ideal in B formed by those elements in B which are mapped into $\mathfrak{P}$ by i) for $\mathfrak{P} \in Spec(A)$. If $\mathfrak{P} \in Spec(A)$ and $\mathfrak{Q} = f(\mathfrak{P})$, there exists a unique homomorphism $i_{\mathfrak{P}}: B_{\mathfrak{Q}} \to A_{\mathfrak{P}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & A \\ \downarrow & & \downarrow \\ B_{\mathfrak{Q}} & \xrightarrow[i_{\mathfrak{P}}]{} & A_{\mathfrak{P}} \end{array}$$

where $B \to B_{\mathfrak{Q}}$ and $A \to A_{\mathfrak{P}}$ denote the canonical homomorphisms associated with the localizations. Similarly, if $\mathfrak{Q} \in Spec(B)$, there exists a homomorphism $i_{\mathfrak{Q}}: B_{\mathfrak{Q}} \to A_{\mathfrak{Q}}$, uniquely induced by i and the localizations, where $A_{\mathfrak{Q}}$ denotes the ring of quotients of A with respect to the multiplicatively closed subset $B - \mathfrak{Q}$ of B.

For our convenience, a Noetherian ring will be said to be *unmixed* if all the associated prime ideals of zero have the same corank (which is equal to the rank of the ring), and an ideal in it will be said to be *unmixed* if it has

---

* Received February 23, 1960.

no imbedded prime divisors. If A' is a Noetherian ring and $\mathfrak{A}'$ is an ideal in A', the unmixedness of $A'/\mathfrak{A}'$ is stronger than that of $\mathfrak{A}'$ itself. But, if A' satisfies the chain condition of prime ideals, an ideal $\mathfrak{A}'$ of rank r and generated by r elements ($r \geqq 0$) is unmixed if and only if $A'/\mathfrak{A}'$ is unmixed;[1] we know that if A' is an algebra of finite type over a field and is unmixed, then it satisfies the chain condition of prime ideals.

One of the theorems proved by W. Krull on the parameter-specialization of ideals in a polynomial ring may be stated as follows:

THEOREM OF KRULL.[2] *Let $B = k[u]$ with a field k and a variable u over k, and let us assume that $i: B \to A$ is of finite type. If B does not contain any zero-divisors of A and A_{B^*} is unmixed for $B^* = B - (0)$, then $A_{\mathfrak{Q}}/\mathfrak{Q}A_{\mathfrak{Q}}$ is unmixed and has the same rank as A_{B^*} (or, equivalently, the principal ideal $\mathfrak{Q}A_{\mathfrak{Q}}$ is unmixed and of rank one) for almost all (i.e., all but a finite number of) prime ideals $\mathfrak{Q}$ in B.*

A. Seidenberg generalized the theorem of Krull into the following form:

THEOREM OF SEIDENBERG.[3] *Let B be an arbitrary polynomial ring over a field, and let us assume that $i: B \to A$ is of finite type. If B does not contain any zero-divisors of A and A_{B^*} is unmixed, there exists a non-empty open subset U of Spec(B) such that, for every $\mathfrak{Q} \in U$, $A_{\mathfrak{Q}}/\mathfrak{Q}A_{\mathfrak{Q}}$ is unmixed and has the same rank as A_{B^*}.*

The above condition on $\mathfrak{Q} \in U$ is equivalent to the following:[4]

---

[1] *Krull's theorem: All the isolated prime divisors of an ideal with r generators in a Noetherian ring have ranks not greater than r.* (Krull [2], and also M. Nagata, "Basic theorems on general commutative rings," *Memoirs of the college of science, Kyoto University*, vol. 24, No. 1 (1955)).

[2] Krull [3].

[3] Seidenberg [4], Appendix.

[4] Let k be the field of coefficients of B. Let K be the field of quotients of B. If $\mathfrak{p}$ is any of the associated prime ideals of zero in A, $\mathfrak{p}A_{B^*}$ is an associated prime ideal of zero in A_{B^*} because B does not contain any zero-divisor of A. Since A_{B^*} is unmixed, $\operatorname{corank}(\mathfrak{p}A_{B^*}) = \operatorname{rank}(A_{B^*})$. Noting that A is a k-algebra of finite type and A_{B^*} is a K-algebra of finite type, we get

$$\operatorname{corank}(\mathfrak{p}) = \dim_k(A/\mathfrak{p}) = \dim_k(B) + \dim_K(A_{B^*}) = \operatorname{rank}(B) + \operatorname{rank}(A_{B^*}).$$

Hence A is unmixed and $\operatorname{rank}(A) = \operatorname{rank}(B) + \operatorname{rank}(A_{B^*})$. Let $\mathfrak{Q}$ be any prime ideal in B such that $\mathfrak{Q}A_{\mathfrak{Q}} = A_{\mathfrak{Q}}$, and $\mathfrak{P}$ any prime ideal in A such that $\mathfrak{P}A_{\mathfrak{Q}}$ is a maximal ideal containing $\mathfrak{Q}A_{\mathfrak{Q}}$. Then $\dim_k(A/\mathfrak{P}) = \dim_k(B/\mathfrak{Q})$, hence, $\operatorname{corank}(\mathfrak{P}) = \operatorname{corank}(\mathfrak{Q})$. Therefore $\operatorname{rank}(A_{\mathfrak{P}}) = \operatorname{rank}(A) - \operatorname{corank}(\mathfrak{P}) = (\operatorname{rank}(B) + \operatorname{rank}(A_{B^*})) - \operatorname{corank}(\mathfrak{Q}) = \operatorname{rank}(\mathfrak{Q}) + \operatorname{rank}(A_{B^*})$. Now, $A_{\mathfrak{Q}}/\mathfrak{Q}A_{\mathfrak{Q}}$ is unmixed and of rank $= \operatorname{rank}(A_{B^*})$ if and only if $\mathfrak{Q}A_{\mathfrak{P}}$ $(= (x_1, \cdots, x_s)A_{\mathfrak{P}})$ is unmixed and of rank $= \operatorname{rank}(A_{\mathfrak{P}}) - \operatorname{rank}(A_{B^*})$ $(= s)$ for every $\mathfrak{P}$ as above if and only if $\mathfrak{Q}A_{\mathfrak{Q}}$ $(= (x_1, \cdots, x_s)A_{\mathfrak{Q}})$ is unmixed and of rank s.

(*) *if* $x_1, \cdots, x_s$ *is a regular system of parameters of* $B_{\mathfrak{Q}}$, *then* $(x_1, \cdots, x_s)A_{\mathfrak{Q}}$ *is unmixed and of rank* s,

and also to the following:

(**) *there exists a prime ideal* $\mathfrak{P}$ *in* A *such that* $\mathfrak{Q} = i^{-1}(\mathfrak{P})$, *and, for every such* $\mathfrak{P}$, $(x_1, \cdots, x_s)A_{\mathfrak{P}}$ *is unmixed and of rank* s *if* $x_1, \cdots, x_s$ *is a regular system of parameters of* $B_{\mathfrak{Q}}$, *where* $A_{\mathfrak{P}}$ *is view as a* $B_{\mathfrak{Q}}$*-algebra by means of* $i_{\mathfrak{P}}$.

We shall consider the following condition on $\mathfrak{P} \in Spec(A)$, which is stronger and more useful than the condition (**) viewed as a condition on a fixed $\mathfrak{P}$, in particular for $t = 0$, that is:

$(***)_t$ (*for each non-negative integer* t).[5] *Let* $\mathfrak{Q} = i^{-1}(\mathfrak{P})$. *For every non-negative integer* $t' \leqq t$, *if* $x_1, \cdots, x_s$ *is any finite system of elements* $(s \geqq 0)$ *in* $B_{\mathfrak{Q}}$ *such that* $\operatorname{rank}(x_1, \cdots, x_s)B_{\mathfrak{Q}} = s$ *and if* $y_1, \cdots, y_{t'}$ *is any system of elements in* $A_{\mathfrak{P}}$ *such that* $\operatorname{rank}(x_1, \cdots, x_s, y_1, \cdots, y_{t'})A_{\mathfrak{P}} = \operatorname{rank}(x_1, \cdots, x_s)A_{\mathfrak{P}} + t'$, *then the ideal* $(x_1, \cdots, x_s, y_1, \cdots, y_{t'})A_{\mathfrak{P}}$ *is unmixed and of* rank $s + t'$.

In connection with this $(***)_t$, we shall also consider the following condition on a Noetherian ring A':

$(-)_t$ (*for each non-negative integer* t). *For every non-negative integer* $t' \leqq t$, *if* $y_1, \cdots, y_{t'}$ *is any system of elements in* A' *such that*

$$\operatorname{rank}(y_1, \cdots, y_{t'})A' = t',$$

then the ideal $(y_1, \cdots, y_{t'})A'$ *is unmixed.*

In particular, the condition $(-)_0$ is equivalent to saying that the zero ideal of A' is unmixed, and $(-)_1$ is known as one of the important conditions for A' to be normal.

We state (and shall prove in § 2) the main result in this paper in the following form:

MAIN THEOREM. (*Generalized theorem of Krull-Seidenberg*). *Let us assume that* B *is a homomorphic image of a regular ring and that* $i: B \to A$ *is of finite type* (*in the sense explained in the very beginning*). *Then there*

[5] Throughout this paper, we adopt the following conventions: A finite system of elements, written as $x_1, \cdots, x_s$, denotes the empty system (i.e., the empty set) if $s = 0$; the empty set of elements generates the zero ideal.

exists an open subset V_t of $Spec(A)$, for each non-negative integer t, such that $\mathfrak{P} \in V_t$ if and only if $\mathfrak{P}$ satisfies the condition $(***)_t$.

The homomorphism $i: B \to A$ induces a continuous mapping $f: Spec(A) \to Spec(B)$, which is called a *morphism* together with the family of homomorphisms $i_{\mathfrak{P}}: B_{f(\mathfrak{P})} \to A_{\mathfrak{P}}$ with $f(\mathfrak{P}) = i^{-1}(\mathfrak{P})$. One knows the following *theorem of C. Chevalley:*[6] *if $i: B \to A$ is of finite type, then the morphism $f: Spec(A) \to Spec(B)$ it induces maps every constructible subset of $Spec(A)$ onto a constructible subset of $Spec(B)$*; in general, a subset of a topological space is said to be *constructible* if it is a finite union of locally closed subsets, i.e., intersections of closed subsets and open subsets.

Let B and $i: B \to A$ be as in the main theorem. Let U'_t be the maximal open subset of $Spec(B)$ that is contained in the image $f(V_t)$ of the open subset V_t of $Spec(A)$ under the induced morphism $f: Spec(A) \to Spec(B)$. Let U''_t be the complement in $Spec(B)$ of the closure of the image $f(Spec(A) - V_t)$ of the closed complement of V_t in $Spec(A)$. We define the open subset V_t of $Spec(B)$ as the intersection of these U'_t and U''_t.

Corollary to Main Theorem. *If there exists an isolated prime ideal $\mathfrak{Q}'$ associated with zero in B such that $A_{\mathfrak{Q}'}$ satisfies the condition $(-)_t$, then the open subset U_t of $Spec(B)$, obtained as above, is not empty (in fact, contains $\mathfrak{Q}'$) and satisfies the following condition: For every $\mathfrak{Q} \in U_t$ there exists at least one $\mathfrak{P} \in Spec(A)$ such that $i^{-1}(\mathfrak{P}) = \mathfrak{Q}$, and if $\mathfrak{P} \in Spec(A)$ is such that $i^{-1}(\mathfrak{P}) \in U_t$, then $\mathfrak{P}$ satisfies the condition $(***)_t$; furthermore, if $\mathfrak{Q} \in U_t$ and if $x_1, \cdots, x_r$ is any system of parameters of $B_{\mathfrak{Q}}$, $A_{\mathfrak{Q}}/(x_1, \cdots, x_r)A_{\mathfrak{Q}}$ satisfies the condition $(-)_t$.*

The corollary is immediate from the main theorem and the definition of the open subset U_t; in fact, we can see that U_t is the maximal open subset of $Spec(B)$ which has the property stated in the corollary. In case $t = 0$, the corollary states a better generalization of the theorem of Krull than that of Seidenberg. The corollary for $t = 1$ is also proved by A. Seidenberg, under certin auxiliary conditions, in the course of his proof of the fact that almost all the hyperplane sections of a normal variety are also normal.[7]

We shall assume and make use of the whole statement of the following theorem due to M. Auslander:

---

[6] Chap. I, § 7, Grothendieck and Dieudonné [1]; Théorème 3. Also Cartan-Chevalley Seminar, 1955-56.

[7] See Seidenberg [4].

THEOREM 0. (*Theorem of Auslander*).[8] *Let A be a homomorphic image of a regular ring. Let $d(A_{\mathfrak{P}}) = \operatorname{rank}(A_{\mathfrak{P}}) - \operatorname{codh}(A_{\mathfrak{P}})$ for each $\mathfrak{P} \in Spec(A)$. Then, for each non-negative integer d, there exists a closed subset F_d of $Spec(A)$ such that $\mathfrak{P} \in F_d$ if and only if $d(A_{\mathfrak{P}}) \geqq d$.*

The above integer $\operatorname{codh}(A_{\mathfrak{P}})$ can be defined as follows: A system of elements in a local ring A', say $x_1, \cdots, x_s$, is defined as *a prime sequence of* A' (or, simply, *an A'-sequence*) if x_i is not a zero divisor in $A'/(x_1, \cdots, x_{i-1})A'$ for $s = 1, \cdots, s$, and if $(x_1, \cdots, x_s)A' \neq A'$ at the same time. An A'-sequence $x_1, \cdots, x_s$ is said to be *maximal* if $x_1, \cdots, x_s, y$ is not an A'-sequence for any element y in A'. Then one can prove:

(′) *all the maximal A'-sequence have the same length.*[9]

The integer $\operatorname{codh}(A')$ is, by definition, the length of any (hence, all the) maximal A'-sequences.

More generally, if S is any subset of a local ring A', a system of elements in S is called *an A'-sequence in S* if it is an A'-sequence. An A'-sequence in S, say $x_1, \cdots, x_s$, is said to be *maximal* if $x_1, \cdots, x_s, y$ is not an A'-sequence for any $y \in S$. As is easily seen, an A'-sequence in S, $x_1, \cdots, x_s$, is maximal if and only if at least one of the associated prime ideals of $(x_1, \cdots, x_s)A'$ contains $S \cap M'$, where M' denotes the maximal ideal of A'. Let $x_1, \cdots, x_s$ be any maximal A'-sequence in S. Let $\mathfrak{P}'$ be an associated prime ideal of $(x_1, \cdots, x_s)A'$ which contains $S \cap M'$. Then $x_1, \cdots, x_s$ is a maximal $A'_{\mathfrak{P}'}$-sequence and every A'-sequence in S is an $A'_{\mathfrak{P}'}$-sequence. Hence, by (′), s is the maximal integer attained by the lengths of A'-sequences in S. It follows that:

(″) *all the A'-sequences in S have the same length, for every subset S of A'.*

---

[8] This theorem was communicated by his letter to the author early in 1958. The letter included a complete and simple proof of the theorem, which was based on a characterization of $\operatorname{codh}(A_{\mathfrak{P}})$ (or, more generally, of homological codimension of an $A_{\mathfrak{P}}$-*module* of finite type) in terms of Ext-functors. His method als gives a proof of the following statement: *Let us assume that A is a homomorphic image of a regular ring and that all the maximal chains of prime ideals in the regular ring have the same length (for example, A is a k-algebra of finite type with a field k), then the set of those $\mathfrak{P}$ such that $\operatorname{corank}(\mathfrak{P}) + \operatorname{codh}(A_{\mathfrak{P}}) \leqq d$ is a closed subset of $Spec(A)$ for each non-negative integer d.* One can see that this conclusion is stronger than that of Theorem 0. Consequently, Theorem 1 can be restated and proved likewise.

For the proof of Theorem 0, a paper by M. Auslander (forthcoming in *Illinois Journal of Mathematics*).

[9] Prop. 3.4, Auslander and Buchsbaum [5]; also [6].

We define $\operatorname{codh}(A', S)$ as the length of any (hence, all the) maximal A'-sequences in S. In this paper we shall freely use the following convention: For an arbitrary set S (not necessarily a subset of A'), if there is given a mapping $i: S \to A'$ (or, every element of S operates on A' like multiplication by a certain element in A') and if there occurs no ambiguity about it by the context, then we write $\operatorname{codh}(A', S)$ instead of $\operatorname{codh}(A', i(S))$.[10]

We shall first prove the following generalization of the Theorem 0 (See § 2):

THEOREM 1. *Let us assume that B is a homomorphic image of a regular ring and that $i: B \to A$ is of finite type. Then, for each non-negative integer d, there exists a closed subset G_d of $Spec(A)$ such that $\mathfrak{P} \in G_d$ if and only if $d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) \geqq d$ with $\mathfrak{Q} = i^{-1}(\mathfrak{P})$, where*

$$d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = \operatorname{rank}(B_{\mathfrak{Q}}) - \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$$

by definition.

We shall see that the integer $d(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$ is always non-negative (Lemma 2 in § 2). Therefore *the set of those $\mathfrak{P} \in Spec(A)$ for which $d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = 0$ is an open subset of $Spec(A)$*. This is the only part of Theorem 1 that we shall use later (in the proof of Main Theorem). We shall see in § 2 that Main Theorem is an immediate consequence of this result and Theorem 0.

Previously $(***)_t$ for non-negative integers t are introduced as conditions on $\mathfrak{P} \in Spec(A)$ for the given B and B-algebra A. More generally, however, they may be viewed as conditions on a given pair of a local ring B' and a local B'-algebra A' (in the places of $B_{\mathfrak{Q}}$ and $A_{\mathfrak{P}}$ respectively).

As will be seen by Lemma 4 of § 2, the condition: $d(A', B') = 0$ for a given pair of a local ring B' and a local B'-algebra A' is equivalent to saying that

$(***)_{-1}$ *every finite system of elements of B', say s elements, which generates an ideal of rank s in B', is an A'-sequence.*

Furthermore, for the sake of convenience, as $(***)_{-d}$ will be regarded the condition: $d(A', B') < d$, for each positive integer d. Theorem 1 will then be considered as a part of Main Theorem by saying that:

Main Theorem is true for all integers t, non-negative and negative, and $(***)_t$ is stronger than $(***)_{t'}$ if $t > t'$; consequently, the corollary to Main Theorem is also true for all integers t (where the condition $(-)_t$ with $t < 0$ is empty).

---

[10] An analogous convention will be used in saying that a system of elements in S is (or is not) an A'-sequence.

In § 3, we shall prove the following proposition, which is not new:

PROPOSITION. *Let B' be a local ring and A' a local B'-algebra. Let $\mathfrak{P}'$ be an ideal in A' such that B' and the B'-algebra $A'/\mathfrak{P}'$ satisfy the condition* $(***)_{-1}$. *Let $x_1, \cdots, x_s$ be any system of parameters of B' and set $\bar{A}' = A'/(x_1, \cdots, x_s)A'$. Then, for an ideal $\mathfrak{P}'_0$ in A', $\mathfrak{P}'_0 \subseteq \mathfrak{P}'$ and $\mathfrak{P}'_0\bar{A}' = \mathfrak{P}'\bar{A}'$ imply $\mathfrak{P}'_0 = \mathfrak{P}'$.*

Main Theorem, in virtue of this proposition, can be proved also useful to study the property of complete intersection (local or even global) of cycles in an algebraic family. As for cycles[11] of codimension one, for example, we shall prove the following

THEOREM 2. *Let V be a normal variety and $\{X_t\}_{t \in T}$ an algebraic family of V-cycles of codimension one with a non-singular parameter variety T. Then there exists a non-empty open subset T' of T (in Zariski topology) such that either all or none of the V-cycles X_t with $t \in T'$ are locally principal (everywhere) on V.*

This theorem will be proved as an application of the main theorem (or rather, of its corollary) for $t = 0$. (Some different proofs of this theorem have been suggested to the author by A. Grothendieck and also by C. S. Seshadri.)

Theorem 2 will be used in a forthcoming paper concerning the notion of algebraic equivalence and the theory of blowing-up (factorization of birational morphisms); in the paper we shall define a certain algebraic equivalence of *Ideals* (= coherent algebraic sheaves of fractional ideals) on an algebraic model. We shall also see certain relations between the two notions of algebraic equivalences, the one defined for arbitrary cycles of codimension one (in the sense of A. Weil)[10] and the other for *divisors in Cartier's sense* (= locally principal cycles of codim. 1) introduced by C. Chevalley.[12] For example, the following results will be included: (In the terminology of Cartier-Chevalley)

((1)) *Let V be a normal variety (defined over a fixed algebraically closed field). Let $G(V)$ denote the additive group of divisors on V, and $G_a(V)$ the subgroup of $G(V)$ of those which are algebraically equivalent to*

[11] See Weil [7], comments on equivalence relations of algebraic cycles. Also, to study the Picard variety of a normal variety from this point of view, S. Lang's book, *Abelian Varieties* (Interscience, New York (No. 7)), is complete.

[12] See a paper of C. Chevalley, "On the Picard variety of a normal variety," forthcoming in the *American Journal of Mathematics* (1959-60).

zero on V. Let V'' be an open subset of V such that $V - V''$ has codimension at least two. Then we have a monomorphism r of $G(V)$ into $G(V'')$ (restriction of divisors to V'') which have the following properties: $G_a(V) \subseteq r^{-1}(G_a(V''))$ (—clear—) and the quotient $r^{-1}(G_a(V''))/G_a(V)$ is a finite group.

((2)) *Let us consider the pairs (V', f) of a normal variety V' and a porper birational morphism $f: V' \to V$ (i.e., a birational transformation of V' onto V which is regular everywhere on V' and such that V' is complete over V with respect to f). If $\mathfrak{F}$ is a coherent algebraic sheaf of fractional ideals on V, $f^*(\mathfrak{F})$ shall denote the coherent algebraic sheaf of fractional ideals on V' generated by the ideals of $\mathfrak{F}$, where the function fields of V and V' are canonically identified by means of f. Let V'' denote the open subset of V which consists of all the simple points of V. Then, given a coherent sheaf $\mathfrak{F}$ of fractional ideals on V, there exists a pair (V', f) such that $f^*(\mathfrak{F})$ is locally principal on V' and is associated with a divisor which is algebraically equivalent to zero on V', if and only if $\mathfrak{F}$ itself is locally principal on V and is associated with a divisor belonging to $r^{-1}(G_a(V''))$.*

((3)) *Let V and (V', f) be as above. Let V'' denote the open subset of V' (also, of V) which consists of those points of V' at which f is biregular. Let r' denote the homomorphism of $G(V')$ into $G(V'')$ (restriction of divisors to V''). Then the homomorphism $G_a(V')/G_l(V') \to G_a(V'')/G_l(V'')$ is injective, where $G_l(V')$ (resp. $G_l(V'')$) denotes the subgroup of principal (linearly equivalent to zero) divisors on V' (resp. V'').*

These theorems (especially ((2))) will play important roles in the study on the lattice of normal projective factors of a birational morphism.

2. Proofs of Theorem 1 and Main Theorem. Let B' be a local ring and A' a local B'-algebra. We shall consider the following condition on A' over B':

(!) *For every ideal $\mathfrak{B}$ in B',* $\operatorname{rank}(\mathfrak{B}) = \operatorname{rank}(\mathfrak{B}A')$.

It is necessary for A to satisfy the condition (!) over B' that A' *dominates* B', i.e., the maximal ideal of B' is the complete inverse image of the maximal ideal of A' under the canonical homomorphism of B' into A'. One can easily prove that if A' is a local ring of an integral extension of a polynomial ring over B', and if A' dominates B', then A' satisfies the condition (!) over B'.

LEMMA 1. *Let B be a Noetherian ring and let $i: B \to A$ be of finite*

type. Then there can be found a finite set of prime ideals, $\mathfrak{P}^j$, in A which possesses the following property: for every $\mathfrak{P} \in Spec(A)$, there exists at least one $\mathfrak{P}^j$ such that $\mathfrak{P}^j \subseteq \mathfrak{P}$ and $A_{\mathfrak{P}}/\mathfrak{P}^j A_{\mathfrak{P}}$ satisfies the condition (!) over $B_{\mathfrak{Q}}/\mathfrak{Q}^j B_{\mathfrak{Q}}$, where $\mathfrak{Q} = i^{-1}(\mathfrak{P})$ and $\mathfrak{Q}^j = i^{-1}(\mathfrak{P}^j)$.

Proof. The proof is by induction on $n = \text{rank}(A)$. If $\text{rank}(A) = 0$, then A has only a finite number of prime ideals and there is nothing to prove. Let us assume that $\text{rank}(A) = n > 0$ and that the lemma is true for $\text{rank}(A) < n$. First take all the isolated prime ideals belonging to zero in A, say $\mathfrak{P}_0^j$. For each j, set $A/\mathfrak{P}_0^j = A^j$ and $B/\mathfrak{Q}_0^j = B^j$, where $\mathfrak{Q}_0^j = i^{-1}(\mathfrak{P}_0^j)$. Then the homomorphism i induces $i^j\colon B^j \to A^j$, which is injective and of finite type. By the normalization theorem, there exists a non-zero element b_j in B^j such that $A^j[b_j^{-1}]$ is integral over a polynomial ring over $B^j[b_j^{-1}]$. Take then all the isolated prime ideals belonging to $b_j A + \mathfrak{P}_0^j$, say $\mathfrak{P}^{jk}$. By the induction assumption, there can be found a finite set of prime ideals $\bar{\mathfrak{P}}^{jkm}$ in $A/\mathfrak{P}^{jk}$, for each $\mathfrak{P}^{jk}$, which possesses the property, stated in the lemma, for $A/\mathfrak{P}^{jk}$ viewed as B-algebra in a natural way. Let $\mathfrak{P}^{jkm}$ be the prime ideal in A such that $\mathfrak{P}^{jkm} \supset \mathfrak{P}^{jk}$ and $\mathfrak{P}^{jkm}/\mathfrak{P}^{jk} = \bar{\mathfrak{P}}^{jkm}$. Then the finite set $\{\mathfrak{P}_0^j, \mathfrak{P}^{jk}, \mathfrak{P}^{jkm}\}$ possesses the property for A as B-algebra.

LEMMA 2. *Let B and $i\colon B \to A$ be as in Lemma 1. If $\mathfrak{P}$ and $\mathfrak{P}'$, $\in Spec(A)$, are such that $\mathfrak{P} \supseteq \mathfrak{P}'$, then always*

$$d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) \geqq d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) \geqq 0,$$

where $\mathfrak{Q} = i^{-1}(\mathfrak{P})$ and $\mathfrak{Q}' = i^{-1}(\mathfrak{P}')$.

Proof. It is clear from the definition of rank and corank that

(1) $\quad \text{rank}(B_{\mathfrak{Q}}) - \text{rank}(B_{\mathfrak{Q}'}) \geqq \text{corank}(\mathfrak{Q}' B_{\mathfrak{Q}})$.

Therefore, to prove the first inequality in the lemma, it is sufficient to prove

(2) $\quad \text{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) - \text{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) \leqq \text{corank}(\mathfrak{Q}' B_{\mathfrak{Q}})$,

where $\text{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'})$ denotes $\text{codh}(A_{\mathfrak{P}'}, i_{\mathfrak{P}'}(B_{\mathfrak{Q}'}))$ and likewise $\text{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$. We can prove (2) by induction on $\text{corank}(\mathfrak{Q}' B_{\mathfrak{Q}}) = s$. If $s = 0$ then $\mathfrak{Q} = \mathfrak{Q}'$, $B_{\mathfrak{Q}} = B_{\mathfrak{Q}'}$ and $A_{\mathfrak{P}'} = (A_{\mathfrak{P}})_{\mathfrak{P}' A_{\mathfrak{P}}}$ (the ring of quotients of $A_{\mathfrak{P}}$ with respect to the prime ideal $\mathfrak{P}' A_{\mathfrak{P}}$). Hence, every $A_{\mathfrak{P}}$-sequence in $B_{\mathfrak{Q}}$ is an $A_{\mathfrak{P}'}$-sequence in $B_{\mathfrak{Q}'}$, which shows $\text{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) \leqq \text{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'})$ and the inequality (2) for $s = 0$. Let us assume that $s > 0$ and that (2) is verified for $\text{corank}(\mathfrak{Q}' B_{\mathfrak{Q}}) < s$. To prove the inequality for $\mathfrak{P}$, $\mathfrak{P}'$ with $\text{corank}(\mathfrak{Q}' B_{\mathfrak{Q}}) = s$, let us take a maximal $A_{\mathfrak{P}}$-sequence in $\mathfrak{Q}' B_{\mathfrak{Q}}$, say $x_1, \cdots, x_t$. As is easily seen, $x_1, \cdots, x_t$ is an $A_{\mathfrak{P}'}$-sequence and its image in $B_{\mathfrak{Q}'}$ is an $A_{\mathfrak{P}'}$-sequence in $B_{\mathfrak{Q}'}$. Therefore

(3) $\operatorname{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) \geqq t.$

If $x_1, \cdots, x_t$ is a maximal $A_{\mathfrak{P}}$-sequence in $B_{\mathfrak{Q}}$, i.e., $\operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = t$, then the left hand side of (2) is not positive by (3) and the inequality (2) is obvious. Hence we may proceed to the case in which we have an element x_{t+1} in $B_{\mathfrak{Q}}$ such that $x_1, \cdots, x_t, x_{t+1}$ is an $A_{\mathfrak{P}}$-sequence in $B_{\mathfrak{Q}}$. There exists an associated prime ideal of $(x_1, \cdots, x_t, x_{t+1})A_{\mathfrak{P}}$ which contains $\mathfrak{Q}'B_{\mathfrak{Q}}$, say $\mathfrak{P}''A_{\mathfrak{P}}$ with $\mathfrak{P}'' \in Spec(A)$. In fact, if it were not the case, there would be an element y in $\mathfrak{Q}'B_{\mathfrak{Q}}$ such that $x_1, \cdots, x_t, x_{t+1}, y$ is an $A_{\mathfrak{P}}$-sequence; so that $x_1, \cdots, x_t, y$ would be an $A_{\mathfrak{P}}$-sequence in $\mathfrak{Q}'B_{\mathfrak{Q}}$, in view of the following result:

$(')_1$ *In a local ring, a finite number of elements form a prime sequence in every ordering if they do in any ordering.*[13]

On the other hand there should not be such y by the selection of $x_1, \cdots, x_t$, and we can conclude the existence of such $\mathfrak{P}''$. Now, let $\mathfrak{Q}'' = i^{-1}(\mathfrak{P}'')$. $\mathfrak{Q}'' \subseteq \mathfrak{Q}$ and $\mathfrak{Q}''B_{\mathfrak{Q}} \supseteq (\mathfrak{Q}', x_{t+1})B_{\mathfrak{Q}}$. But x_{t+1} is not in $\mathfrak{Q}'B_{\mathfrak{Q}}$ because $x_1, \cdots, x_t$ is maximal as an $A_{\mathfrak{P}}$-sequence *in* $\mathfrak{Q}'B_{\mathfrak{Q}}$ and, at the same time $x_1, \cdots, x_t, x_{t+1}$ is an $A_{\mathfrak{P}}$-sequence. It follows that $\operatorname{corank}(\mathfrak{Q}''B_{\mathfrak{Q}}) \leqq \operatorname{corank}(\mathfrak{Q}'B_{\mathfrak{Q}}) - 1$. So, by the induction assumption applied to $\mathfrak{P}$ and $\mathfrak{P}''$, we get

(4) $\operatorname{corank}(\mathfrak{Q}'B_{\mathfrak{Q}}) - 1 \geqq \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) - \operatorname{codh}(A_{\mathfrak{P}''}, B_{\mathfrak{Q}''}),$

where $\mathfrak{Q}'' = i^{-1}(\mathfrak{P}'')$.

Since $\mathfrak{P}''A_{\mathfrak{P}}$ is an associated prime ideol of $(x_1, \cdots, x_t, x_{t+1})A_{\mathfrak{P}}$, $x_1, \cdots, x_t, x_{t+1}$ is a maximal $A_{\mathfrak{P}''}$-sequence and, of course, a maximal $A_{\mathfrak{P}''}$-sequence in $B_{\mathfrak{Q}''}$. It shows that

(5) $\operatorname{codh}(A_{\mathfrak{P}''}, B_{\mathfrak{Q}''}) = t + 1.$

The inequality (2) is now immediate as follows:

$$\begin{aligned} \operatorname{corank}(\mathfrak{Q}'B_{\mathfrak{Q}}) &\geqq \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) - \operatorname{codh}(A_{\mathfrak{P}''}, B_{\mathfrak{Q}''}) + 1 && \text{(by (4))} \\ &= \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) - t && \text{(by (5))} \\ &\geqq \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) - \operatorname{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) && \text{(by (3)).} \end{aligned}$$

The first inequality is now proved. The second one is an immediate consequence of the first. In fact, if $\mathfrak{P}^*$ is an associated prime ideal of zero in A which is contained in $\mathfrak{P}'$, then

$$d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) \geqq d(A_{\mathfrak{P}^*}, B_{\mathfrak{Q}^*}) = \operatorname{rank}(B_{\mathfrak{Q}^*}) \geqq 0,$$

where $\mathfrak{Q}^* = i^{-1}(\mathfrak{P}^*)$.

---

[13] Cor. 2.9 of Prop. 2.8, Auslander and Buchsbaum [6].

We shall consider a chain condition of the following type, on the prime ideals in a Noetherian ring B':

(*The restricted chain condition*). *For every pair of prime ideals* $\mathfrak{Q}$ *and* $\mathfrak{Q}'$ *in* B' *such that* $\mathfrak{Q} \subseteq \mathfrak{Q}'$, *all the maximal chains of strictly ascending prime ideals between* $\mathfrak{Q}$ *and* $\mathfrak{Q}'$ *have the same length.*

It is clear that if the condition is satisfied in B', then so in every homomorphic image of B', and that, for example, it is satisfied in every homomorphic image of a regular ring.

LEMMA 3. *Let us assume that* B *satisfies the restricted chain condition on the prime ideals (in the above sense), that* $i: B \to A$ *is of finite type, and that* A *is a homomorphic image of a regular ring. Let* $\mathfrak{P}$ *be any prime ideal in* A *and* $\mathfrak{Q} = i^{-1}(\mathfrak{P})$. *Then there exists an open subset* V *of* $Spec(A)$ *such that* $\mathfrak{P} \in V$ *and* $d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) = d(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$ *for every* $\mathfrak{P} \subset \mathfrak{P}' \in V$ *and* $\mathfrak{Q}' = i^{-1}(\mathfrak{P}')$.

Proof. First we shall reduce the proof to the case in which $\operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = 0$. To do it, we take a maximal $A_{\mathfrak{P}}$-sequence in B (which is necessarily such one in $B_{\mathfrak{Q}}$), say $x_1, \cdots, x_q$ with $q = \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$. Replacing A by $A[u^{-1}]$ with a suitable $u \in A - \mathfrak{P}$, we may assume that $x_1, \cdots, x_q$ is an A-sequence in B. If the lemma is true in case $\operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = 0$, there exists an open subset $\bar{V}$ of $Spec(\bar{A})$, $\bar{A}$ denoting $A/(x_1, \cdots, x_q)A$ which is viewed as a B-algebra in the natural way, such that $\bar{\mathfrak{P}} = \mathfrak{P}/(x_1, \cdots, x_q)A$ is in $\bar{V}$ and $d(\bar{A}_{\bar{\mathfrak{P}}'}, B_{\mathfrak{Q}'}) = d(\bar{A}_{\bar{\mathfrak{P}}}, B_{\mathfrak{Q}})$ for every $\bar{\mathfrak{P}} \subseteq \bar{\mathfrak{P}}' \in \bar{V}$ and $\bar{\mathfrak{P}}' = \mathfrak{P}'/(x_1, \cdots, x_q)A$. Take an open subset V of $Spec(A)$ which induces the open subset $\bar{V}$ in $Spec(\bar{A})$; for example, the subset consisting of those $\mathfrak{P}' \in Spec(A)$ such that either $\mathfrak{P}' \not\supseteq (x_1, \cdots, x_q)A$ or $\mathfrak{P}'/(x_1, \cdots, x_q)A = \bar{\mathfrak{P}}' \in \bar{V}$. This open subset V of $Spec(A)$ satisfies the condition stated in the lemma. In fact, if $\mathfrak{P} \subset \mathfrak{P}' \in V$, then $\bar{\mathfrak{P}} \subset \bar{\mathfrak{P}}' \in \bar{V}$ and therefore

$$\begin{aligned} d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) &= \operatorname{rank}(B_{\mathfrak{Q}'}) - \operatorname{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) \\ &= \operatorname{rank}(B_{\mathfrak{Q}'}) - (\operatorname{codh}(\bar{A}_{\bar{\mathfrak{P}}'}, B_{\mathfrak{Q}'}) + q) = d(\bar{A}_{\bar{\mathfrak{P}}'}, B_{\mathfrak{Q}'}) - q \\ &= d(\bar{A}_{\bar{\mathfrak{P}}}, B_{\mathfrak{Q}}) - q = \operatorname{rank}(B_{\mathfrak{Q}}) - (\operatorname{codh}(\bar{A}_{\bar{\mathfrak{P}}}, B_{\mathfrak{Q}}) + q) \\ &= \operatorname{rank}(B_{\mathfrak{Q}}) - \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}). \end{aligned}$$

The reduction of the proof is now done, and from now on we assume that $\operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = 0$. First, let us take an element $u \in B - \mathfrak{Q}$ which is contained in all but those contained in $\mathfrak{Q}$ of the associated prime ideals of zero in B. Let V' be the open subset of $Spec(A)$ which is defined by the ideal uA.

Then

(6) $\mathfrak{P} \in V'$, *and* $\operatorname{rank}(B_{\mathfrak{Q}'}) - \operatorname{rank}(B_{\mathfrak{Q}}) = \operatorname{corank}(\mathfrak{Q}B_{\mathfrak{Q}'})$ *if* $\mathfrak{P} \subset \mathfrak{P}' \in V'$ *and* $\mathfrak{Q}' = i^{-1}(\mathfrak{P}')$.

The equality in (6) follows from the assumption that B satisfies the restricted chain condition on the prime ideals, because $\mathfrak{Q}$ contains all of those associated prime ideals of zero in B which are contained in $\mathfrak{Q}'$. Next, we apply Theorem 0 in § 1 to A. Let F_d be the closed subsets of $Spec(A)$ and take the isolated prime ideals $\mathfrak{P}_{dk}$ of F_d for every d. We have only a finite number of such $\mathfrak{P}_{dk}$ because F_d are empty for all $d > \operatorname{rank}(A)$. Then we apply Lemma 1 to the B-algebra $A/\mathfrak{P}_{dk}$, for each $\mathfrak{P}_{dk}$, and choose a finite set of prime ideals $\mathfrak{P}_{dk^j}$ in A such that all the $\mathfrak{P}_{dk^j}$ contain $\mathfrak{P}_{dk}$ and, if $\mathfrak{P}_{dk} \subseteq \mathfrak{P}'' \in Spec(A)$, $A_{\mathfrak{P}''}/\mathfrak{P}_{dk^j}A_{\mathfrak{P}''}$ satisfies the condition (!) over $B_{\mathfrak{Q}''}/\mathfrak{Q}_{dk^j}B_{\mathfrak{Q}''}$ for at least one one $\mathfrak{P}_{dk^j}$, where $\mathfrak{Q}_{dk^j} = i^{-1}(\mathfrak{P}_{dk^j})$. When $\mathfrak{P}'' \in Spec(A)$ is given, we can choose $\mathfrak{P}_{dk}$ such that $\mathfrak{P}_{dk} \subseteq \mathfrak{P}''$ and $d(A_{\mathfrak{P}''}) = d(A_{\mathfrak{P}_{dk}})$, hence $d(A_{\mathfrak{P}''}) = d(A_{\mathfrak{P}_{dk^j}})$ for every $\mathfrak{P}_{dk^j} \subseteq \mathfrak{P}''$. Therefore, denoting by $\mathfrak{P}^*$ any one of the $\mathfrak{P}_{dk^j}$ (all d, k and j), we can state:

(7) *There exists a finite set of prime ideals* $\mathfrak{P}^*$ *in* A *which has the following property: for every* $\mathfrak{P}'' \in Spec(A)$, *there can be found at least one* $\mathfrak{P}^*$ *such that* $\mathfrak{P}'' \supseteq \mathfrak{P}^*$, $d(A_{\mathfrak{P}''}) = d(A_{\mathfrak{P}^*})$ *and* $A_{\mathfrak{P}''}/\mathfrak{P}^*A_{\mathfrak{P}''}$ *satisfies the condition* (!) *over* $B_{\mathfrak{Q}''}/\mathfrak{Q}^*B_{\mathfrak{Q}''}$ *with* $\mathfrak{Q}^* = i^{-1}(\mathfrak{P}^*)$ *and* $\mathfrak{Q}'' = i^{-1}(\mathfrak{P}'')$.

Let V'' be the intersection of those open subsets of $Spec(A)$ which are defined by the ideals $\mathfrak{P}^* \not\subseteq \mathfrak{P}$. Since we have only a finite number of $\mathfrak{P}^*$, V'' is an open subset of $Spec(A)$. Clearly $\mathfrak{P} \in V''$.

(8) *If* $\mathfrak{P}'' \in V''$, *then* $\mathfrak{P}'' \supseteq \mathfrak{P}^*$ *implies* $\mathfrak{P} \supseteq \mathfrak{P}^*$.

We can now prove that the open subset $V = V' \cap V''$ of $Spec(A)$, containing $\mathfrak{P}$, possesses the property stated in the lemma. Let $\mathfrak{P} \subset \mathfrak{P}' \in V$ and $\mathfrak{Q}' = i^{-1}(\mathfrak{P}')$. We want to prove that $d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) = d(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$. Let us take a maximal $A_{\mathfrak{P}'}$-sequence in $B_{\mathfrak{Q}'}$, $x_1, \cdots, x_s$. By the maximality, there exists an associated prime ideal of $(x_1, \cdots, x_s)A_{\mathfrak{P}'}$ which contains the maximal ideal of $B_{\mathfrak{Q}'}$, say $\mathfrak{P}''A_{\mathfrak{P}'}$ with $\mathfrak{P}'' \in Spec(A)$. Then $\mathfrak{Q}'' = i^{-1}(\mathfrak{P}'') = \mathfrak{Q}'$, and $x_1, \cdots, x_s$ is a maximal $A_{\mathfrak{P}''}$-sequence, hence,

(9) $\operatorname{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) = s = \operatorname{codh}(A_{\mathfrak{P}''})$.

Let us take and fix one $\mathfrak{P}^*$, for $\mathfrak{P}''$, having the property stated in (7). Since $\mathfrak{P}'$ is in V''', $\mathfrak{P}''$ is also in V''. By (8), $\mathfrak{P}^* \subseteq \mathfrak{P}$. Therefore if $\mathfrak{Q}^* = i^{-1}(\mathfrak{P}^*)$, $\mathfrak{Q}^* \subseteq \mathfrak{Q}$ and $\operatorname{corank}(\mathfrak{Q}^*B_{\mathfrak{Q}''}) = \operatorname{corank}(\mathfrak{Q}^*B_{\mathfrak{Q}'}) \geqq \operatorname{corank}(\mathfrak{Q}B_{\mathfrak{Q}'})$. By (7),

$d(A_{\mathfrak{P}''}) = d(A_{\mathfrak{P}^\bullet})$, i.e., $\operatorname{rank}(A_{\mathfrak{P}'}) - \operatorname{codh}(A_{\mathfrak{P}''}) = \operatorname{rank}(A_{\mathfrak{P}^\bullet}) - \operatorname{codh}(A_{\mathfrak{P}^\bullet})$. Hence

$$\begin{aligned}\operatorname{codh}(A_{\mathfrak{P}''}) &\geqq \operatorname{codh}(A_{\mathfrak{P}''}) - \operatorname{codh}(A_{\mathfrak{P}^\bullet}) \\ &= \operatorname{rank}(A_{\mathfrak{P}''}) - \operatorname{rank}(A_{\mathfrak{P}^\bullet}) \geqq \operatorname{corank}(\mathfrak{P}^* A_{\mathfrak{P}''}).\end{aligned}$$

Again by (7) $A_{\mathfrak{P}''}/\mathfrak{P}^* A_{\mathfrak{P}''}$ satisfies the condition (!) over $B_{\mathfrak{Q}''}/\mathfrak{Q}^* B_{\mathfrak{Q}''}$ (See the paragraph preceding Lemma 1), hence,

$$\operatorname{rank}(A_{\mathfrak{P}''}/\mathfrak{P}^* A_{\mathfrak{P}''}) \geqq \operatorname{rank}(B_{\mathfrak{Q}''}/\mathfrak{Q}^* B_{\mathfrak{Q}''}),$$

i.e., $\operatorname{corank}(\mathfrak{P}^* A_{\mathfrak{P}''}) \geqq \operatorname{corank}(\mathfrak{Q}^* B_{\mathfrak{Q}''})$. We have seen that $\operatorname{corank}(\mathfrak{Q}^* B_{\mathfrak{Q}''})$ $\operatorname{corank}(\mathfrak{Q} B_{\mathfrak{Q}'})$ and that $\operatorname{codh}(A_{\mathfrak{P}''}) \geqq \operatorname{corank}(\mathfrak{P}^* A_{\mathfrak{P}''})$. We therefore get (by these three inequalities)

$$(10) \quad \operatorname{codh}(A_{\mathfrak{P}''}) \geqq \operatorname{corank}(\mathfrak{Q} B_{\mathfrak{Q}'}).$$

Thus $\operatorname{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) - \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = \operatorname{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'})$

$$\begin{aligned}&\text{(because } \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = 0 \text{ by assumption)} \\ &= \operatorname{codh}(A_{\mathfrak{P}''}) && \text{(by (9))} \\ &\geqq \operatorname{corank}(\mathfrak{Q} B_{\mathfrak{Q}'}) && \text{(by (10))} \\ &= \operatorname{rank}(B_{\mathfrak{Q}'}) - \operatorname{rank}(B_{\mathfrak{Q}}).\end{aligned}$$

The last equality is by (6); we took $\mathfrak{P}'$ such that $\mathfrak{P} \subseteq \mathfrak{P}' \in V \subseteq V'$. The result can be written as

$$\operatorname{rank}(B_{\mathfrak{Q}'}) - \operatorname{codh}(A_{\mathfrak{P}'}, B_{\mathfrak{P}'}) \leqq \operatorname{rank}(B_{\mathfrak{Q}}) - \operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$$

or, $d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) \leqq d(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$. By Lemma 2 we can conclude the equality $d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) = d(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$.[14]

THEOREM* 1.[15] *Let us assume that B satifies the restricted chain condition on the prime ideals, that $i: B \to A$ is of finite type and that A is a homomorphic image of a regular ring. Then, for each non-negative integer d, there exists a closed subset G_d of $Spec(A)$ such that $\mathfrak{P} \in G_d$ if and only if $d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) \geqq d$ with $\mathfrak{Q} = i^{-1}(\mathfrak{P})$.*

[14] Notice that the inclusion relation of $\mathfrak{P}$ and $\mathfrak{P}'$ (and consequently the inequality in the conclusion) was reversed in Lemma 2.

[15] The assumption of Theorem* 1 is weaker than that of Theorem 1 announced in the introduction (§ 1). In fact, if B is a homomorphic image of a regular ring, then it satisfies the restricted chain condition, and, moreover, if A is a B-algebra cf finite type, then A is also a homomorphic image of a regular ring, because a polynomial ring over a regular ring is also regular. (Furthermore, it can be seen that Theorem* 1 remains true even without the restricted chain condition in B.)

Proof. Let us fix a non-negative integer d. Let us denote by V_d the set of those $\mathfrak{P} \in Spec(A)$ such that $d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) < d$ with $\mathfrak{Q} = i^{-1}(\mathfrak{P})$. The theorem amounts to saying that V_d is an open subset of $Spec(A)$. Denoting by V_d' *the maximal* open subset contained in V_d (which exists because an arbitrary union of open subsets is open), we shall prove that $V_d' = V_d$. Suppose $V_d' \neq V_d$. Let us take an *isolated* prime ideal $\mathfrak{P}$ in $V_d - V_d'$. First, we see that $\mathfrak{P}$ is isolated even in $Spec(A) - V_d'$. In fact, if $\mathfrak{P} \supseteq \mathfrak{P}' \in Spec(A) - V_d'$, then, by Lemma 2, $\mathfrak{P}' \in V_d$, hence, $\in V_d - V_d'$. Therefore $\mathfrak{P}$, isolated in $V_d - V_d'$ coincides with $\mathfrak{P}'$, i.e., $\mathfrak{P}$ is isolated in $Spec(A) - V_d'$. Since V_d' is open, there exist only a finite number of isolated prime ideals in $Spec(A) - V_d'$, say $\mathfrak{P}_1 = \mathfrak{P}, \mathfrak{P}_2, \cdots, \mathfrak{P}_p$. Let V' be the open subset of $Spec(A)$ defined by the ideal $\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_p$ (or, the whole $Spec(A)$ if $p = 1$). Clearly $\mathfrak{P} \in V'$. On the other hand, there exists an open subset V of $Spec(A)$ such that $\mathfrak{P} \in V$ and $d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) = d(A_{\mathfrak{P}}, B_{\mathfrak{Q}})$ if $\mathfrak{P} \subset \mathfrak{P}' \in V$. (See Lemma 3). Let $V'' = V \cap V'$. Then V'' is an open subset of $Spec(A)$ containing $\mathfrak{P}$. What is more, V'' is contained in V_d. In fact, let $\mathfrak{P}' \in V''$. If $\mathfrak{P}' \supseteq \mathfrak{P}$, then $\mathfrak{P}' \in V$ implies that $d(A_{\mathfrak{P}'}, B_{\mathfrak{Q}'}) = d(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) < d$, hence, $\mathfrak{P}' \in V_d$. If $\mathfrak{P}' \not\supseteq \mathfrak{P}$, $\mathfrak{P}'$ does not contain any of the isolated prime ideal in $Spec(A) - V_d'$, i.e., it is contained in V_d', hence, in V_d. Thus $V'' \cup V_d'$ is an open subset contained in V_d and containing $\mathfrak{P} \notin V_d'$. This contradicts the maximality of V_d', and we can conclude that $V_d' = V_d$, i.e., V_d is open. We get G_d as the complement of V_d in $Spec(A)$.

MAIN THEOREM*.[16] *Let B and $i: B \to A$ be as in the above theorem. Then, for each non-negative integer t, there exists an open subset V_t of $Spec(A)$ such that $\mathfrak{P} \in V_t$ if and only if $\mathfrak{P}$ satisfies the condition $(***)_t$.* (See § 1.)

Proof. The statement and the proof of (7), given in the proof of Lemma 3, are independent of the given $\mathfrak{P}$ and the additional assumption: $\operatorname{codh}(A_{\mathfrak{P}}, B_{\mathfrak{Q}}) = 0$. We adopt it here in the same form as it is. Let V_t^* be the open subset of $Spec(A)$ which is the intersection of those open subsets of $Spec(A)$ defined by the ideals $\mathfrak{P}^*$ not satisfying $(***)_t$. It is clear from the definition of $(***)_t$ that if $\mathfrak{P} \supseteq \mathfrak{P}^*$ and if $\mathfrak{P}^*$ does not satisfy $(***)_t$, then $\mathfrak{P}$ does not $(***)_t$. Hence it is necessary for $\mathfrak{P}$ to satisfy $(***)_t$ that $\mathfrak{P} \in V_t^*$. First we shall prove that if $\mathfrak{P} \in V_t^*$, then

(11) *for an $A_{\mathfrak{P}}$-sequence $x_1, \cdots, x_s$ in $B_{\mathfrak{Q}}$ and t' $(\leqq t)$ elements*

[16] See the footnote 15. (One can also prove Main Theorem* without assuming the restricted chain condition in B.)

$y_1, \cdots, y_{t'}$ *in* $A_{\mathfrak{P}}$ *such that* $\operatorname{rank}(x_1, \cdots, x_s, y_1, \cdots, y_{t'})A_{\mathfrak{P}} = s + t'$, *the ideal* $(x_1, \cdots, x_s, y_1, \cdots, y_{t'})A_{\mathfrak{P}}$ *is unmixed.*

In proving it, we may assume that $x_1, \cdots, x_s, y_1, \cdots, y_{t'}$ is an $A_{\mathfrak{P}}$-sequence; in fact, if otherwise, there would exist t'' $(0 \leqq t'' \leqq t' - 1)$ such that $x_1, \cdots, x_s, y_1, \cdots, y_{t''}$ is an $A_{\mathfrak{P}}$-sequence but the ideal $(x_1, \cdots, x_s, y_1, \cdots, y_{t''})A_{\mathfrak{P}}$ is not unmixed. Let $\mathfrak{P}''$ be any prime ideal in A such that $\mathfrak{P}''A_{\mathfrak{P}}$ is an associated prime ideal of $(x_1, \cdots, x_s, y_1, \cdots, y_{t'})A_{\mathfrak{P}}$. We want to prove that $\operatorname{rank}(A_{\mathfrak{P}''}) = s + t'$. Let us take and fix a $\mathfrak{P}^*$ which has the properties stated in (7) for the above $\mathfrak{P}''$. If $d(A_{\mathfrak{P}^*})$ is proved to be zero, then $d(A_{\mathfrak{P}''}) = d(A_{\mathfrak{P}^*}) = 0$ (by (7)) and therefore $\operatorname{rank}(A_{\mathfrak{P}''}) = \operatorname{codh}(A_{\mathfrak{P}''}) = s + t'$. (See that $x_1, \cdots, x_s, y_1, \cdots, y_{t'}$ is a maximal $A_{\mathfrak{P}''}$-sequence.) We shall prove that $d(A_{\mathfrak{P}^*}) = 0$. First of all, the following is clear:

(12) $\operatorname{codh}(A_{\mathfrak{P}''}) = s + t' \leqq \operatorname{codh}(A_{\mathfrak{P}''}, B_{\mathfrak{Q}''}) + t'$.

Since $A_{\mathfrak{P}''}/\mathfrak{P}^*A_{\mathfrak{Q}''}$ satisfies the condition (!) over $B_{\mathfrak{Q}''}/\mathfrak{Q}^*B_{\mathfrak{Q}''}$ (see (7)),

(13) $\operatorname{corank}(\mathfrak{P}^*A_{\mathfrak{P}''}) \geqq \operatorname{corank}(\mathfrak{Q}^*B_{\mathfrak{Q}''})$.

Since $d(A_{\mathfrak{P}''}) = d(A_{\mathfrak{P}^*})$ (see (7)), we get

$$\operatorname{codh}(A_{\mathfrak{P}''}) - \operatorname{codh}(A_{\mathfrak{P}^*}) = \operatorname{rank}(A_{\mathfrak{P}''}) - \operatorname{rank}(A_{\mathfrak{P}^*}) \geqq \operatorname{corank}(\mathfrak{P}^*A_{\mathfrak{P}''}),$$

hence,

$$\begin{aligned}
(14)\quad \operatorname{codh}(A_{\mathfrak{P}^*}) &\leqq \operatorname{codh}(A_{\mathfrak{P}''}) - \operatorname{corank}(\mathfrak{P}^*A_{\mathfrak{P}''}) \\
&\leqq \operatorname{codh}(A_{\mathfrak{P}''}, B_{\mathfrak{Q}''}) + t' - \operatorname{corank}(\mathfrak{P}^*A_{\mathfrak{P}''}) \quad \text{by (12)} \\
&\leqq \operatorname{codh}(A_{\mathfrak{P}''}, B_{\mathfrak{Q}''}) + t' - \operatorname{corank}(\mathfrak{Q}^*B_{\mathfrak{Q}''}) \quad \text{by (13)} \\
&\leqq \operatorname{codh}(A_{\mathfrak{P}^*}, B_{\mathfrak{Q}^*}) + t' \quad \text{by (2)}.
\end{aligned}$$

On the other hand, $\mathfrak{P} \in V_t^*$ and $\mathfrak{P} \supseteq \mathfrak{P}'' \supseteq \mathfrak{P}^*$. Therefore, by the definition of V_t^*, $\mathfrak{P}^*$ satisfies the condition $(***)_t$. Hence, if $\operatorname{rank}(A_{\mathfrak{P}^*}) > \operatorname{rank}(B_{\mathfrak{Q}^*}) + t$, then $\operatorname{codh}(A_{\mathfrak{P}^*}) \geqq \operatorname{rank}(B_{\mathfrak{Q}^*}) + t + 1$, which is impossible by the above result. Therefore $\operatorname{rank}(A_{\mathfrak{P}^*}) \leqq \operatorname{rank}(B_{\mathfrak{Q}^*}) + t$, and then $(***)_t$ says that $\operatorname{codh}(A_{\mathfrak{P}^*}) = \operatorname{rank}(A_{\mathfrak{P}^*})$, i.e., $d(A_{\mathfrak{P}^*}) = 0$. Thus the assertion (11) on the open subset V_t^* is proved. Let G_1 be the closed subset of $Spec(A)$ obtained in Theorem 1, and let $V_1' = Spec(A) - G_1$. As is easily seen, if $\mathfrak{P}$ satisfies $(***)_t$, $\mathfrak{P} \in V_1'$. Setting $V_t = V_1' \cap V_t^*$, we can prove that $\mathfrak{P} \in V_t$ if and only if $\mathfrak{P}$ satisfies the condition $(***)_t$. In fact, the if part has been already proved, and the only-if part follows from the following lemma, in virtue of the above result (11).

LEMMA 4. *Let B' be a local ring and A' a local B'-algebra dominating B'; namely, A' is a local ring with a homomorphism $i: B' \to A'$ under which the maximal ideal of B' is the complete inverse image of that of A'. If* $\operatorname{codh}(A', B') = \operatorname{rank}(B')$ (i.e., $d(A', B') = 0$), *then every finite system of elements in B', which generates an ideal in B' of rank = the length of the system, is an A'-sequence.*

Proof. The proof will be an immediate consequence of the results (") of § 1 and $(')_1$ cited in the proof of Lemma 2. Let $x_1, \cdots, x_s$ be any finite system of elements of B' such that $\operatorname{rank}(x_1, \cdots, x_s)B' = s$. Then we can extend it to a system of parameters of B', say $x_1, \cdots, x_s, x_{s+1}, \cdots, x_m$. Let $S = \{x_1, \cdots, x_m\}$. If $x_{1'}, \cdots, x_{m'}$ is a maximal A'-sequence in S, then one of the associated prime ideals of $(x_{1'}, \cdots, x_{m'})A'$ contains S (more rigorously, $i(S)$). Hence it also contains the image of the maximal ideal of B', which implies that $x_{1'}, \cdots, x_{m'}$ is a maximal A'-sequence in B'. However, $m = \operatorname{rank}(B') = \operatorname{codh}(A', B')$, by assumption, and therefore $x_{1'}, \cdots, x_{m'}$ contains all the elements in S. By $(')_1$, $x_1, \cdots, x_m$ is an A'-sequence, hence obviously, $x_1, \cdots, x_s$ is an A'-sequence.

3. Proofs of Proposition and Theorem 2. First, as for the proposition stated in § 1, the existence of B' is not essential and we restate and prove it as follows:

PROPOSITION. *Let A' be a local ring and $\mathfrak{J}'$ an ideal in A'. Let $x_1, \cdots, x_s$ be a system of elements in A', which is an $A'/\mathfrak{J}'$-sequence. For an ideal $\mathfrak{J}'_0$ in A', $\mathfrak{J}'_0\bar{A}' = \mathfrak{J}'_0\bar{A}'$ and $\mathfrak{J}'_0 \subseteq \mathfrak{J}'$ imply $\mathfrak{J}'_0 = \mathfrak{J}'$, where $\bar{A}' = A'/(x_1, \cdots, x_s)A'$.*

Proof. The proof will be by induction on s. First consider the case in which $s = 1$. What is asserted in this case is: *If x_1 is neither a unit of A' nor a zero-divisor in $A'/\mathfrak{J}'$, then $\mathfrak{J}' + x_1A' = \mathfrak{J}'_0 + x_1A'$ and $\mathfrak{J}'_0 \subseteq \mathfrak{J}'$ imply $\mathfrak{J}'_0 = \mathfrak{J}'$.* Since A' is a local ring and x_1 is not a unit of A' we have $\mathfrak{J}'_0 = \bigcap_{n=1}^{\infty} (\mathfrak{J}'_0 + x_1^n A')$. Therefore it is sufficient to prove that $\mathfrak{J}' \subseteq \mathfrak{J}'_0 + x_1^n A'$ for all positive integers n. This can be proved by induction on n as follows: For $n = 1$, it is included in the assumption. Then, assume that $\mathfrak{J}'_0 + x_1^{n-1}A' \supseteq \mathfrak{J}'$ for any $n > 1$. Let y be any element of $\mathfrak{J}'$. Then $y = y_0 + x_1^{n-1}z$ with $y_0 \in \mathfrak{J}'_0$ and $z \in A'$. Then $z \in (\mathfrak{J}' : x_1^{n-1}A')$. But, since x_1 is not a zero -divisor in $A'/\mathfrak{J}'$, we have $(\mathfrak{J}' : x_1^{n-1}A') = \mathfrak{J}'$. Hence, by the assumption in Proposition, $z = y'_0 + x_1z'$ with $y'_0 \in \mathfrak{J}'_0$ and $z' \in A'$, and therefore

$$y = y_0 + x_1^{n-1}z = (y_0 + y'_0) + x_1^n z' \in \mathfrak{Z}'_0 + x_1^n A'.$$

Thus the proposition for $s = 1$ is proved. Assume that $s > 1$ and that the proposition is proved for smaller s. Taking $\mathfrak{Z}' + x_1A'$, $\mathfrak{Z}'_0 + x_1A'$ and $x_2, \cdots, x_s$ in the places of $\mathfrak{Z}'$, $\mathfrak{Z}'_0$ and $x_1, \cdots, x_s$, we can easily check that the induction assumption is applicable to them and asserts the equality $\mathfrak{Z}' + x_1A' = \mathfrak{Z}'_0 + x_1A'$. By the above result for $s = 1$, $\mathfrak{Z}' = \mathfrak{Z}'_0$ follows.

Let $\boldsymbol{V}$ be a normal variety, defined over an algebraically closed field k. Let $\boldsymbol{T}$ be a non-singular variety, defined over k, and $\boldsymbol{Y}$ a $\boldsymbol{V} \times \boldsymbol{T}$-cycle of codimension one. We know that the cycle $\boldsymbol{X}_t = \mathrm{pr}_{\boldsymbol{V}}[\boldsymbol{Y} \cdot \boldsymbol{V} \times \boldsymbol{t}]$ on $\boldsymbol{V}$ is defined for almost all $\boldsymbol{t} \in \boldsymbol{T}$, i.e., for all $\boldsymbol{t}$ in a certain non-empty open subset of $\boldsymbol{T}$. We want to prove that *there exists a non-empty open subset $\boldsymbol{T}'$ of $\boldsymbol{T}$ such that:* 1) *$\boldsymbol{X}_t$ is defined for every $\boldsymbol{t} \in \boldsymbol{T}'$ and* 2) *either all or none of the X_t with $\boldsymbol{t} \in \boldsymbol{T}'$ are locally principal* (*i.e., divisors on $\boldsymbol{V}$ in Cartier's sense*). For this assertion we may assume that $\boldsymbol{T}$ is affine, without any loss of generality. Also, without any loss of generality, we may assume that $\boldsymbol{V}$ is affine, by the fact that a variety admits a finite covering of affine open subsets, and then that the cycle $\boldsymbol{Y}$ is positive, because one can find a principal cycle $\boldsymbol{Y}_0$ of codimension one on the affine variety $\boldsymbol{V} \times \boldsymbol{T}$ such that $\boldsymbol{Y} + \boldsymbol{Y}_0$ is positive.

We can now formulate the problem in the following generality. Let T be a regular integral domain, and W a T-algebra of finite type such that $i: T \to W$ is injective. Let $\mathfrak{Z}_0$ be an ideal in the rings of quotients V_0 of W with respect to $T^* = T = (0)$. We assume that *all the associated prime ideals of $\mathfrak{Z}_0$ have the same rank, say $r > 0$.* Then the intersection $\mathfrak{Z} = \mathfrak{Z}_0 \cap W$ has the same property. If $\mathfrak{t}$ is a prime ideal in T such that $\mathrm{rank}(\mathfrak{Z}, \mathfrak{t})W_{\mathfrak{t}} = \mathrm{rank}(\mathfrak{t}) + r$, then we set $V_{\mathfrak{t}} = W_{\mathfrak{t}}/\mathfrak{t}W_{\mathfrak{t}}$ and $\mathfrak{Z}_{\mathfrak{t}} = \mathfrak{Z}V_{\mathfrak{t}}$ $(= (\mathfrak{Z}, \mathfrak{t})W_{\mathfrak{t}}/\mathfrak{t}W_{\mathfrak{t}})$ and we say that *the specialization $\mathfrak{Z}_{\mathfrak{t}}$ of $\mathfrak{Z}_0$ is defined.* One can prove that if $\mathfrak{Z}_{\mathfrak{t}}$ is defined, then $\mathrm{rank}\,(\mathfrak{Z}_{\mathfrak{t}}) \geqq r$. (Note: if $s = \mathrm{rank}(\mathfrak{t})$, $\mathfrak{t}W_{\mathfrak{t}}$ is generated by s elements because T is a regular ring.) If $\mathfrak{Z}_{\mathfrak{t}}$ is defined, we also define and denote by $\mathfrak{Z}(\mathfrak{t})$ *the reduced specialization of $\mathfrak{Z}_0$ at* $\mathfrak{t}$ as follows: $\mathfrak{Z}(\mathfrak{t})$ *= the intersection of those primary components of $\mathfrak{Z}_{\mathfrak{t}}$ which belong to prime ideals of rank r.*

Remark. Let $\boldsymbol{V}$, $\boldsymbol{T}$ and $\boldsymbol{Y}$ be a normal variety, a non-singular variety and a positive $\boldsymbol{V} \times \boldsymbol{T}$-cycle of codimension one respectively. Let k be a field over which $\boldsymbol{V}$ and $\boldsymbol{T}$ are defined and $\boldsymbol{Y}$ is rational. Let T be an affine ring ring over $\boldsymbol{T}$ over k, and W such of $\boldsymbol{V} \times \boldsymbol{T}$ which contains T. Let $\mathfrak{Z}$ be the ideal of $\boldsymbol{Y}$ in W. Let $\boldsymbol{t}$ be a point of $\boldsymbol{T}$ and $\mathfrak{t}$ the ideal of $\boldsymbol{t}$ in T, assuming that $\boldsymbol{t}$ is contained in the open subset of $\boldsymbol{T}$ with which the ring T is associated.

The ring of quotients V_0 of W with respect to $T-(0)$ is an affine ring of $\boldsymbol{V}$, and $\mathfrak{J}V_0$ is the ideal of a *generic* cycle of the family $\{\boldsymbol{X}_t\}$ with reference to field k. Set $\mathfrak{J}_0=\mathfrak{J}V_0$. Then we can see that the cycle $\boldsymbol{X}_t$ is defined if and only if $\mathfrak{J}_0\cap W_t=\mathfrak{J}W_t$ (and $\mathfrak{J}_t$ is defined) for all W as above. (The second condition in the parenthesis follows from the first, because $\operatorname{rank}(\mathfrak{J})=1$.) Furthermore, it is easy to see that there exists an affine ring T of $\boldsymbol{T}$ such that $\mathfrak{J}_0\cap W=\mathfrak{J}$ for all W containing T, hence, if $\mathfrak{t}$ is a prime ideal in T, then $\mathfrak{J}_0\cap W_{\mathfrak{t}}=\mathfrak{J}W_{\mathfrak{t}}$ for all W as above. Let $\boldsymbol{t}$ and $\mathfrak{t}$ be as above. Suppose $\boldsymbol{X}_t$ (hence $\mathfrak{J}_{\mathfrak{t}}$) be defined. Then *the reduced specialization* $\mathfrak{J}(\mathfrak{t})$ *is the ideal of the cycle* $\boldsymbol{X}_t$ *in the affine ring* $V_{\mathfrak{t}}$ *of* $\boldsymbol{V}$.

In general, an ideal $\mathfrak{J}'$ in a Noetherian ring W' will be said to be *locally principal* if $\operatorname{rank}(\mathfrak{J}')=r>0$ and $\mathfrak{J}'$ is generated by r elements.

THEOREM 2. *Let T be a regular integral domain, W a T-algebra of finite type such that $i\colon T\to W$ is injective and $\mathfrak{J}_0$ an ideal in the ring of quotients V_0 of W with respect to $T-(0)$. Let us assume that all the isolated prime ideals in $W_{\mathfrak{P}}$ have the same corank for every prime ideal $\mathfrak{P}$ in W and that all the associated prime ideals of $\mathfrak{J}_0$ have the same rank $r>0$. Let $\mathfrak{J}_{\mathfrak{t}}$ and $\mathfrak{J}(\mathfrak{t})$ be the symbols which were introduced above and called respectively the specialization and the reduced specialization of $\mathfrak{J}_0$ at $\mathfrak{t}\in Spec(T)$. Then there exists a non-empty open subset U of $Spec(T)$ such that:* i) $\mathfrak{J}_{\mathfrak{t}}$ *(hence $\mathfrak{J}(\mathfrak{t})$) are defined for all $\mathfrak{t}\in U$,* ii) $\mathfrak{J}_{\mathfrak{t}}=\mathfrak{J}(\mathfrak{t})$ *for all $\mathfrak{t}\in U$, and* iii) *either all or none of the $\mathfrak{J}_{\mathfrak{t}}$ with $\mathfrak{t}\in U$ are locally principal.*

Proof. Let $\mathfrak{J}=\mathfrak{J}_0\cap W$ and $A=W/\mathfrak{J}$. Then the T-algebra A satisfies the conditions (on A with $B=T$) in the corollary to Main Theorem for $t=0$; namely, beside basic assumptions on T and A, the ring of quotients A_0 of A with respect to $T-(0)$ satisfies the condition $(-)_0$. By the corollary, there exists a non-empty open subset U_0 of $Spec(T)$ such that, every $\mathfrak{t}\in U_0$, $\mathfrak{t}A_{\mathfrak{t}}\neq A_{\mathfrak{t}}$ and a regular system of parameters of $T_{\mathfrak{t}}$ is an $A_{\mathfrak{p}}$-sequence and generates an unmixed ideal in $A_{\mathfrak{p}}$ if $\mathfrak{p}\in Spec(A)$ and $\mathfrak{p}\cap T=\mathfrak{t}$. We shall prove that all $\mathfrak{t}\in U_0$ possess the properties i) and ii). Let $\mathfrak{t}\in U_0$ and take a regular system of parameters, $x_1,\cdots,x_s$, of $T_{\mathfrak{t}}$ ($s=\operatorname{rank}(\mathfrak{t})$). Then the above result implies that: (a) $(\mathfrak{J},\mathfrak{t})W_{\mathfrak{t}}\neq W_{\mathfrak{t}}$ and (b) $(\mathfrak{J},\mathfrak{t})W_{\mathfrak{P}}$ $(=(\mathfrak{J},x_1,\cdots,x_s)W_{\mathfrak{P}})$ is unmixed and $(\mathfrak{J},\mathfrak{t})W_{\mathfrak{P}}/\mathfrak{J}W_{\mathfrak{P}}$ is of rank s for every $\mathfrak{P}\in Spec(W)$ such that $\mathfrak{P}\supseteq\mathfrak{J}$ and $\mathfrak{P}\cap T=\mathfrak{t}$. Since W is a homomorphic image of a regular ring (that is, a polynomial ring over T), it satisfies the *restricted* chain condition on the prime ideals. (See the definition preceding Lemma 3.) By the assumption that all the isolated prime ideals of zero in $W_{\mathfrak{P}}$ have the same corank, $W_{\mathfrak{P}}$ satisfies *the chain condition on the*

prime ideals, i. e., all the maximal chain of prime ideals in $W_{\mathfrak{P}}$ have the same length. In (b), if $\mathfrak{P}$ is an associated prime ideal of $(\mathfrak{J}, \mathfrak{t})$ then $\operatorname{rank}(W_{\mathfrak{P}}/\mathfrak{J}W_{\mathfrak{P}}) = s$ and, by the above chain condition,

$$\operatorname{rank}(W_{\mathfrak{P}}) = s + \operatorname{rank}(\mathfrak{J}W_{\mathfrak{P}}) = \operatorname{rank}(\mathfrak{t}) + r.$$

(Recall the assumption that all the associated prime ideals of $\mathfrak{J}_0$, hence of $\mathfrak{J}$, have the same rank r.) In other words, all the associated prime ideals of $(\mathfrak{J}, \mathfrak{t})W_{\mathfrak{t}}$ have the same $\operatorname{rank} = \operatorname{rank}(\mathfrak{t}) + r$. The statements i) and ii) follow immediately. We now want to find a non-empty open subset U_0 of U for which iii) holds. We shall consider the following two cases separately: one in which $\mathfrak{J}_0$ is locally principal and the other in which $\mathfrak{J}_0$ is not. Suppose $\mathfrak{J}_0$ be locally principal. First of all, we can easily see that there exists an open subset V of $Spec(W)$ such that $\mathfrak{P} \in V$ if and only if either $\mathfrak{J}W_{\mathfrak{P}} = W_{\mathfrak{P}}$ or $\mathfrak{J}W_{\mathfrak{P}}$ is principal, i. e., generated by r elements. By Chevalley's theorem, the image of $Spec(W) - V$ in $Spec(T)$ is a *constructible* subset of $Spec(T)$. Since $\mathfrak{J}_0 = \mathfrak{J}V_0$ is locally principal, this constructible subset of $Spec(T)$ does not contain any open non-empty subset of $Spec(T)$. Therefore there exists a non-empty open subset of $Spec(T)$, say U', which does not intersect with the image of $Spec(W) - V$. It is clear that the non-empty open subset $U_0 \cap U' = U$ is such that, in addition to i) and ii), all the $\mathfrak{J}_{\mathfrak{t}}$ with $\mathfrak{t} \in U$ are locally principal. Let us now consider the other case in which $\mathfrak{J}_0$ is not locally principal. Then there exist at least one prime ideal $\mathfrak{P}$ in W such that $\mathfrak{P} \cap T = (0)$, i. e., $\mathfrak{P}V_0 \in Spec(V_0)$ and that $\mathfrak{J}W_{\mathfrak{P}}$ is not principal, i. e., not generated by r elements, and $\mathfrak{J}W_{\mathfrak{P}} \neq W_{\mathfrak{P}}$. Since $\mathfrak{P} \cap T = (0)$, the canonical homomorphism of T into the T-algebra $W/\mathfrak{P}$ is injective and therefore, again in virtue of the theorem of Chevalley, there can be found a non-empty open subset U'' of $Spec(T)$ which is contained in the image of $Spec(W/\mathfrak{P})$ in $Spec(T)$. Obviously, this open subset U'' has the property that for every $\mathfrak{t}'' \in U''$ there exists at least one $\mathfrak{P}'' \in Spec(W)$ such that $\mathfrak{P}'' \cap T = \mathfrak{t}''$, $\mathfrak{J}W_{\mathfrak{P}''} \neq W_{\mathfrak{P}''}$ and $\mathfrak{J}W_{\mathfrak{P}''}$ is not principal; in fact, we have $\mathfrak{P}'' \in Spec(W)$ such that $\mathfrak{P}'' \cap T = \mathfrak{t}''$ and $\mathfrak{P}'' \supseteq \mathfrak{P} \supseteq \mathfrak{J}$. Let $U = U_0 \cap U''$, which is a non-empty open subset of $Spec(T)$. If $\mathfrak{t}'' \in U$, such $\mathfrak{P}'' \in Spec(W)$ as above possesses moreover the property that a regular system of parameters of $T_{\mathfrak{t}''}$ is an $A_{\mathfrak{p}}$-sequence, where $A = W/\mathfrak{J}$ and $\mathfrak{p} = \mathfrak{P}''/\mathfrak{J}$, hence, a $W_{\mathfrak{P}''}/\mathfrak{J}W_{\mathfrak{P}''}$-sequence. (It follows from the definition of U_0.) By Proposition, where $A' = W_{\mathfrak{P}''}$ and $\mathfrak{J}' = \mathfrak{J}W_{\mathfrak{P}''}$, if $(\mathfrak{J}, \mathfrak{t})W_{\mathfrak{P}''}/\mathfrak{t}W_{\mathfrak{P}''}$ is generated by r elements then $\mathfrak{J}W_{\mathfrak{P}''}$ is also generated by r elements. But this should not be true. Hence $\mathfrak{J}_{\mathfrak{t}''}$ $(= (\mathfrak{J}, \mathfrak{t})W_{\mathfrak{t}}/\mathfrak{t}W_{\mathfrak{t}})$ is not locally principal. (Note: $W_{\mathfrak{P}''}/\mathfrak{t}W_{\mathfrak{P}''}$ is identified

with a local ring of W_t/tW_t.) Again the open subse U of $Spec(T)$ has the properties required in the theorem.

REFERENCES.

[1] J. Dieudonné and A. Grothendieck, *Élements de géometrie algébrique*, forthcoming in mathematical publications of l'Institut des Hautes Edues Scientifiques.

[2] W. Krull, "Primidealketten in allgemeinen Ringbereiche," *Sitzungsberichte der Heidelberger Akademie der Wissenschaften*, vol. 7 (1928), pp. ———.

[3] ———, "Parameterspecializierung in Polynomringen," *Archiv der Mathematik*, Band 1 (1948), pp. 56-64.

[4] A. Seidenberg, "The hyperplane sections of normal varieties," *Transactions of the American Mathematical Society*, vol. 69 (1950), pp. 357-386.

[5] M. Auslander and D. A. Buchsbaum, "Codimension and multiplicity," *Annals of Mathematics*, vol. 68 (1958), pp. 625-657.

[6] ———, "Homological dimension in local rings," *Transactions of the American Mathematical Society*, vol. 85 (1957), pp. 390-405.

[7] A. Weil, *Foundations of algebraic Geometry*, American Mathematical Society Colloquium Publications, 1946.

THE WEDDERBURN DECOMPOSITION OF COMMUTATIVE BANACH ALGEBRAS.*

By WILLIAM G. BADÉ and PHILIP C. CURTIS, JR.[1]

I. Let $\mathfrak{A}$ be a Banach algebra with radical $\mathfrak{R}$. If the dimension of $\mathfrak{A}$ is finite, the Wedderburn principal theorem states that there exists a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{B} + \mathfrak{R}$ and $\mathfrak{B} \cap \mathfrak{R} = \{0\}$. Moreover, if $\mathfrak{A}$ is commutative, $\mathfrak{B}$ is necessarily unique. It is the purpose of this paper to investigate under what conditions this theorem will hold for infinite dimensional commutative Banach algebras. In 1954 Zelinsky [17] gave an example which showed that the Wedderburn principal theorem did not hold in general for infinite dimensional algebras. No topological assumptions were made in this construction, and in particular the algebra is not a Banach algebra. In 1955 Feldman [6] constructed an example of a commutative Banach algebra with one dimensional radical but with no closed subalgebra $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{B} + \mathfrak{R}$ and $\mathfrak{B} \cap \mathfrak{R} = \{0\}$. Another example with similar properties is given by Glaeser [8]. Various condition under which the theorem does hold for closed subalgebras $\mathfrak{B}$ are discussed by both Feldman and Glaeser.

In this paper we shall also be concerned with the validity of the Wedderburn theorem for non-closed subalgebras $\mathfrak{B}$, so the following terminology will be helpful. We shall call a Banach algebra *decomposable* if there exists a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with the property that $\mathfrak{A} = \mathfrak{B} + \mathfrak{R}$ and $\mathfrak{B} \cap \mathfrak{R} = \{0\}$ ($\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$). The decomposition is unique if there is only one such subalgebra. If there exists a closed subalgebra $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$, then $\mathfrak{A}$ will be said to be *strongly decomposable*. A strong decomposition is unique if $\mathfrak{B}$ is unique among closed subalgebras of $\mathfrak{A}$.

Unfortunately, nothing but the obvious relations between these notions holds in general for commutative Banach algebras. In [4]† the present authors showed that Feldman's example was decomposable even though not strongly

* Received February 23, 1960.

[1] This work was supported by the Office of Naval Research under contract Nonr 233 (59), the Air Force Office of Scientific Research under contract SAR/AF-49(636)-153 and by the National Sciences Foundation.

† Erratum to [4]: On page 604, line 11, read $y(\overline{F})$ for $y(F)$.

so. A simple example discussed in § V of this paper shows that a commutative Banach algebra may have a non-unique decomposition but a unique strong decomposition. A more illuminating example, also discussed in § V, shows that a commutative algebra need not be decomposable at all. In fact, this example is isomorphic modulo the radical to the ring of convergent sequences.

On the positive side of the ledger, however, some fairly general criteria can be given under which a commutative algebra is decomposable. Many of the results are valid for mildly non-commutative algebras as well. As might be expected from the finite dimensional situation, most of the results depend on an abundance of idempotents. For convenience we shall always assume that the algebra possesses a unit. The basic criteria for decomposability, proved in § II, is the following: If A has totally disconnected maximal ideal space $\Phi_{\mathfrak{A}}$, then $\mathfrak{A}$ is decomposable and $\mathfrak{A}/\mathfrak{R} \simeq C(\Phi_{\mathfrak{A}})$ [2] if and only if the idempotents of $\mathfrak{A}$ form a bounded set. If the latter condition holds, then $\mathfrak{A}$ is strongly decomposable and the strong decomposition is unique. Actually, if $\mathfrak{A}/\mathfrak{R} \simeq C(\Phi_{\mathfrak{A}})$, we know of no examples for which the decomposition fails to be unique. The existence of a non closed complement to the radical is equivalent to the existence of a discontinuous isomorphism of $C(\Phi_{\mathfrak{A}})$ into $\mathfrak{A}$, and no such examples are known (cf. [4]).

In § III it is proved that the set of idempotents of A will automatically be bounded if the idempotents satisfy a weak interpolation axiom. This axiom will be satisfied if and only if $\Phi_{\mathfrak{A}}$ is a totally disconnected F space in the sense of Gillman and Henriksen. Thus, for such spaces $\Phi_{\mathfrak{A}}$, which include extremally disconnected compact spaces, we can give a virtually complete Wedderburn structure for the algebra $\mathfrak{A}$. More precisely, for a commutative Banach algebra $\mathfrak{A}$, if $\Phi_{\mathfrak{A}}$ is a totally disconnected F space, then $\mathfrak{A}/\mathfrak{R} \simeq C(\Phi_{\mathfrak{A}})$, $\mathfrak{A}$ is strongly decomposable and the strong decomposition is unique. An obvious but interesting corollary is that if Ω is a totally disconnected F space, then $C(\Omega)$ is the only commutative semi-simple Banach algebra with Ω as maximal ideal space, e.g. l_∞ is the only semi-simple Banach algebra having the Čech compactification of the integers as maximal ideal space.

In § IV we investigate conditions on the radical which will insure decomposability of $\mathfrak{A}$ under the assumption that $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$. For an arbitrary compact space Ω, $\mathfrak{A}$ will be strongly decomposable if the dimension of $\mathfrak{R}$ is finite. If Ω is totally disconnected, this will be the case if $\mathfrak{R}$ is nilpotent. In both cases the decomposition is unique. Neither the assumption

[2] The symbol $\simeq$ will always mean an algebraic isomorphism. No topological conditions are assumed.

that $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ or $\mathfrak{R}^n = \{0\}$ alone is sufficient to guarantee decomposability, however. Examples illustrating these facts are discussed in § V.

II. *Let* $\mathfrak{A}$ be a Banch algebra which we will assume possesses a unit denoted by e. This is only a convenience, for if $\mathfrak{A}$ has no unit and $\mathfrak{A}'$ is formed in the usual way by adjoining one to $\mathfrak{A}$, then $\mathfrak{A}$ is (strongly) decomposable iff $\mathfrak{A}'$ is (strongly) decomposable. Further, a decomposition or a strong decomposition is unique in $\mathfrak{A}$ iff it is unique in $\mathfrak{A}'$. We leave the verification of these statements to the reader.

We will denote by $\mathfrak{R}$ the Jacobson radical[3] of $\mathfrak{A}$. $\mathfrak{A}/\mathfrak{R}$ is a semisimple Banach algebra under the usual coset norm $\| \tilde{x} \| = \inf_{r \in \mathfrak{R}} \| x + r \|$, where $\tilde{x} = x + \mathfrak{R}$. Unless otherwise stated this will always be the norm taken in $\mathfrak{A}/\mathfrak{R}$. If $x \in \mathfrak{R}$, then $\lim_{n \to \infty} \| x^n \|^{1/n} = 0$; and if A is commutative, this property characterizes the elements of $\mathfrak{R}$. We will denote by ν the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{R}$. ν is continuous, and if there exists a subalgebra $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$, then ν maps $\mathfrak{B}$ isomorphically onto $\mathfrak{A}/\mathfrak{R}$.

Let ν' be the inverse of the restriction of ν to $\mathfrak{B}$. ν' is an isomorphism of $\mathfrak{A}/\mathfrak{R}$ onto $\mathfrak{B}$, and it is easily verified that ν' is continuous iff $\mathfrak{B}$ is closed. If $\mathfrak{B}$ is closed, then $\mathfrak{A}$ is the topological direct sum of $\mathfrak{B}$ and $\mathfrak{R}$ by the well known theorem of Kober [12], and $\mathfrak{B}$ is homeomorphic with $\mathfrak{A}/\mathfrak{R}$.

Next we collect a few elementary facts from algebra. They are valid in a more general setting, but the following formulation is sufficient for our needs. Although we are concerned primarily with commutative algebras, many of the results of this section will be stated for mildly non-commutative algebras to facilitate their use in § IV.

LEMMA 2.1. *Let* $P_{\mathfrak{A}}$ *denote the set of idempotents of a Banach algebra* $\mathfrak{A}$. *Then* (1) $P_{\mathfrak{A}} \cap R = \{0\}$. (2) *If* $x \equiv x^2(\mathfrak{R})$, *then there exists* $p \in P_{\mathfrak{A}}$, *satisfying* $x \equiv p(\mathfrak{R})$, *and* $yp = py$ *for any* $y \in \mathfrak{A}$ *such that* $yx = xy$. (3) *If* $p, q \in P_{\mathfrak{A}}$, *and* $pq = qp$, *then* $p \equiv q(\mathfrak{R})$ *implies* $p = q$.

Proof. (1) is immediate since $\mathfrak{R}$ consists of quasi regular elements, and 0 is the only quasi regular idempotent. For the proof of (2) we refer the reader to [15, Theorem 2.3.9]. Lastly, if $p, q \in P_{\mathfrak{A}}$ and $pq = qp$, then $p - q \in \mathfrak{R}$ implies $p(p-q), -q(p-q) \in P_{\mathfrak{A}} \cap \mathfrak{R}$. Hence $p(p-q) = 0 = -q(p-q)$ or $p = q$.

[3] For a full discussion of the properties of the radical the reader is referred to [15, chapter 2].

LEMMA 2.2. *If $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ for some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, and if $pq = qp$ for each $p, q \in P_{\mathfrak{A}}$, then $P_{\mathfrak{A}} \subset \mathfrak{B}$.*

Proof. For each $x \in \mathfrak{A}$ there is a unique representation $x = y_x + k$, $y_x \in \mathfrak{B}$, $k \in \mathfrak{R}$. The mapping $\tau: x \to y_x$ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Therefore, if $p \in P_{\mathfrak{A}}$, then $q = \tau(p) \in P_{\mathfrak{A}} \cap \mathfrak{B}$ and $p \equiv q(\mathfrak{R})$. By Lemma 2.1 (3) $p = q$, and thus $P_{\mathfrak{A}} \subset \mathfrak{B}$.

LEMMA 2.3. *If $pr = rp$ for all $p \in P_{\mathfrak{A}}$, $r \in \mathfrak{R}$, and if $\mathfrak{A}/\mathfrak{R}$ is commutative, then $pq = qp$, $p, q \in P_{\mathfrak{A}}$.*

Proof. Let $p, q \in P_{\mathfrak{A}}$. We note first that if $pq \in \mathfrak{R}$, then $pq = qp = 0$. For by hypothesis $pq \in \mathfrak{R}$ implies $pqp = pq$, and hence $(pq)^2 = pq = 0$ by Lemma 2.1(1). Since $\mathfrak{A}/\mathfrak{R}$ is commutative, $pq \in \mathfrak{R}$ implies $qp \in \mathfrak{R}$, and the assertion follows. In general, by Lemma 2.1(2) there is a $q_0 \in P_{\mathfrak{A}}$ such that $pq \equiv q_0 \equiv qp(\mathfrak{R})$. Clearly $(1-p)q_0$ and $(1-q)q_0 \in \mathfrak{R}$. Thus, by what we have just shown, $q_0 = pq_0 = q_0p = qq_0 = q_0q$. Therefore $p - q_0, q - q_0 \in P_{\mathfrak{A}}$, and since $(p-q_0)(q-q_0) \in \mathfrak{R}$, an application of the same arguments proves that $pq = q_0 = qp$.

Before proving the main result of this section we note the following fact which is due to Rickart [14, Theorem 1]. If ν is any isomorphism of $C(\Omega)$[4] into a commutative Banach algebra $\mathfrak{A}$, then each ω may be extended to a multiplicative linear functional on $\mathfrak{A}$. This yields two corollaries of use to us. One, a result of Kaplansky [11, Theorem 6.2], is that $\|\nu(x)\| \geqq \sup_{\omega \in \Omega} |x(\omega)|$. Hence ν is a homeomorphism iff the range of ν is closed. Secondly, if $\mathfrak{A} = C(\Omega')$ and ν is continuous, then ν maps onto $\mathfrak{A}$ if $\nu(C(\Omega))$ separates the points of Ω'. This follows from the Stone-Weierstrass theorem, since under these hypotheses $\nu(C(\Omega))$ must be self adjoint in $C(\Omega')$.

The next result is our main criterion for decomposability.

THEOREM 2.4. *Let $\mathfrak{A}$ be a Banach algebra with radical $\mathfrak{R}$, and let $\mathfrak{B} = \mathfrak{A}/\mathfrak{R}$. Assume that $\mathfrak{B}$ is commutative with totally disconnected maximal ideal space $\Phi_{\mathfrak{B}}$ and assume that $pq = qp$ for $p, q \in P_{\mathfrak{A}}$. Then $P_{\mathfrak{A}}$ is a bounded set* iff *$\mathfrak{A}$ is decomposable and $\mathfrak{B} \simeq C(\Phi_{\mathfrak{B}})$. If $P_{\mathfrak{A}}$ is a bounded set, then $\mathfrak{A}$ is strongly decomposable and the strong decomposition is unique. If in addition $\mathfrak{R}$ is nilpotent, the decomposition is necessarily strong and unique.*

Proof. Suppose $\mathfrak{A}$ is decomposable and $\mathfrak{B} \simeq C(\Phi_{\mathfrak{B}})$. Let $\mathfrak{B}'$ be a subalgebra of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{B}' \oplus \mathfrak{R}$, and let P_1 denote the set of idempotents of $C(\Phi_{\mathfrak{B}})$. Since $\mathfrak{B}' \simeq \mathfrak{B} \simeq C(\Phi_{\mathfrak{B}})$, there exist an isomorphism μ of $C(\Phi_{\mathfrak{B}})$

[4] The symbols Ω, Ω' will always denote compact Hausdorff spaces.

onto $\mathfrak{B}'$. By Lemma 2.2 $P_{\mathfrak{A}} \subset \mathfrak{B}'$, hence $\mu(P_1) = P_{\mathfrak{A}}$. By [4, Theorem 2.3] $P_{\mathfrak{A}}$ is a bounded set in $\mathfrak{A}$.

Conversely, suppose $P_{\mathfrak{A}}$ is bounded. Let $\mathfrak{B}'$ be the closed commutative subalgebra of $\mathfrak{A}$ generated by $P_{\mathfrak{A}}$. By [5, Theorem 17] there exists a bicontinuous isomorphism τ of $\mathfrak{B}'$ onto $C(\Phi_{\mathfrak{B}'})$. It is an obvious corollary of the result of Rickart mentioned earlier [14, Theorem 1] that for $x \in \mathfrak{B}'$, $\sup_{\phi \in \Phi_{\mathfrak{B}'}} |\tau(x)(\phi)| = r_{\mathfrak{B}'}(x)$.[5] From this it follows that $\mathfrak{B}' \cap \mathfrak{R} = \{0\}$, since $x \in \mathfrak{B}' \cap \mathfrak{R}$ implies $\lim_{n\to\infty} \| x^n \|^{1/n} = r_{\mathfrak{B}'}(x) = 0$.

Let ν be the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{B} = \mathfrak{A}/\mathfrak{R}$, and let ρ be the Gelfand isomorphism of $\mathfrak{B}$ into $C(\Phi_{\mathfrak{B}})$. If $\mu = \rho\nu$, μ maps $\mathfrak{A}$ into $C(\Phi_{\mathfrak{B}})$ and is one to one on $\mathfrak{B}'$. We assert μ maps $\mathfrak{B}'$ onto $C(\Phi_{\mathfrak{B}})$. By the well known result of Silov [16],[6] $\rho(P_{\mathfrak{B}}) = P_1$ and by Lemma 2.1(2), $\nu(P_{\mathfrak{A}}) = P_{\mathfrak{B}}$. Therefore, $\mu(P_{\mathfrak{A}}) = P_1$ and $\mu(\mathfrak{B}')$ contains the linear span $\mathfrak{C}$ of P_1. Since $\Phi_{\mathfrak{B}}$ is assumed to be totally disconnected, $\mathfrak{C}$ (and therefore $\mu(\mathfrak{B}')$) is dense in $C(\Phi_{\mathfrak{B}})$. Therefore, by [14, Theorem 1], $\mu\tau^{-1}$ maps $C(\Phi_{\mathfrak{B}'})$ onto $C(\Phi_{\mathfrak{B}})$. Consequently, $\mu(\mathfrak{B}') = C(\Phi_{\mathfrak{B}}) = \rho(\mathfrak{B})$, and $A = \mathfrak{B}' \oplus \mathfrak{R}$.

Finally, suppose $\mathfrak{A} = \mathfrak{B}'' \oplus \mathfrak{R}$ for some subalgebra $\mathfrak{B}''$. There exists an isomorphism λ of $C(\Phi_{\mathfrak{B}})$ onto $\mathfrak{B}''$, and since by Lemma 2.2 $\mathfrak{B} \cap \mathfrak{B}'' \supset P_{\mathfrak{A}}$, it follows that $\lambda(P_1) = P_{\mathfrak{A}}$. If $\mathfrak{B}''$ is closed, λ is a homeomorphism and consequently $\mathfrak{B}' = \mathfrak{B}''$. If $\mathfrak{R}$ is nilpotent[7], then by [4, Theorem 4.5] λ must be continuous, and the theorem follows.

We remark that no examples are known for which the decomposition in the above theorem fails to be unique. As is shown in [4] the problem of constructing such an example is equivalent to the problem of constructing a discontinuous isomorhpism of $C(\Omega)$ for a compact space Ω.

III. In this section we shall suppose that $\mathfrak{A}$ is a Banach algebra in which the elements of $P_{\mathfrak{A}}$ commute. Under this hypothesis it then follows that $P_{\mathfrak{A}}$ will automatically be bounded if it satisfies a simple interpolation axiom. This is an extension of an earlier result of Badé [3, Theorem 2.2] to the effect that a σ-complete Boolean algebra of projections in a Banach space is always bounded. One may strip away the Banach space framework from

---

[5] For a commutative Banach algebra A, and an element $x \in A$,

$$r_A(x) = \sup_{\phi \in \Phi_A} |x(\phi)|.$$

[6] A more accessible reference is [15, Theorem 3.6.3].

[7] Added in proof: The hypothesis of [4, Theorem 4.5] is that $\mathfrak{R}$ is a nil ideal. This involves no more generality than the assumption of nilpotency, for a simple category argument shows that if a commutative Banach algebra is a nil algebra, then it must be nilpotent.

this theorem and obtain a result about general Boolean rings. This we do below except that we weaken σ-completeness to the following interpolation axiom.

Definition. Let A be a Boolean ring. A is said to satisfy property (I) if for any pair of sequences $\{a_n\}, \{b_n\} \subset A$, $n = 1, 2, \cdots$, for which $a_n a_m = b_n b_m = 0$, $n \neq m$, and $a_n b_m = 0$, for all n, m, there exists an element $a \in A$ such that $aa_n = a_n$ and $ab_n = 0$, $n = 1, 2, \cdots$. The following two results give the facts that we need. We do not assume that the Boolean ring possesses a unit, although in the applications this will be the case.

LEMMA 3.1. *Let A be a Boolean ring, ρ a function on A to the non-negative reals with the following property.*

$$(1) \qquad \text{If } x \geqq y, \text{ then } \rho(x - y) \geqq |\rho(x) - \rho(y)|.$$

If ρ is unbounded on A, then there exists a sequence $\{x_n\} \subset A$ such that $x_n x_m = 0$, $n \neq m$, and $\lim_{n \to \infty} \rho(x_n) = \infty$.

Proof. It follows easily from (1) that if $xy = 0$, $\rho(x + y) \leqq \rho(x) + \rho(y)$. An element $x \in A$ is said to have property (α) if $\sup_{y \leqq x} \rho(y) = \infty$. We prove the theorem first under the assumption that there exist an $x \in A$ with (α). We will construct a sequence $\{y_n\} \subset A$, $n = 1, 2, \cdots$, $y_n \geqq y_{n+1}$, satisfying $\rho(y_n) \geqq n + \rho(y_{n-1})$. If $x_n = y_n - y_{n+1}$, then $\{x_n\}$ satisfies the conclusion of the lemma. Note that if $y \leqq x$ and x has (α), then either y or $(x - y)$ has (α). Let $y_1 = x$ and choose $z_1 \leqq y_1$ for which $\rho(z_1) \geqq 2 + 2\rho(y_1)$. Let y_2 be the member of the pair $(z_1, y_1 - z_1)$ with (α). Then $\rho(y_2) \geqq 2 + \rho(y_1)$. Now choose $z_2 \leqq y_2$ for which $\rho(z_2) \geqq 3 + 2\rho(y_2)$. If y_3 is the member of the pair $(z_2, y_2 - z_2)$ with (α), then $\rho(y_3) \geqq 3 + \rho(y_2)$. Proceeding inductively we construct the desired sequence $\{y_n\}$.

Now assume that for each $x \in A$, $\sup_{y \leqq x} \rho(y) < \infty$. We construct the sequence $\{x_n\}$ directly. Choose x_1 with $\rho(x_1) \geqq 1$ and let $M_1 = \sup_{y \leqq x_1} \rho(y)$. Then there exists $z_1 \in A$ with $\rho(z_1) \geqq 2 + M_1$. If $x_2 = z_1 - x_1 z_1$, $x_1 x_2 = 0$, and $\rho(x_2) = \rho(z_1 - x_1 z_1) \geqq \rho(z_1) - \rho(x_1 z_1) \geqq 2$. Let $M_2 = \sup_{y \leqq x_1 + x_2} \rho(y)$. Choose z_2 for which $\rho(z_2) \geqq 3 + M_2$. If $x_3 = z_2 - (x_1 + x_2) z_2$, $x_3 x_2 = x_3 x_1 = 0$ and $\rho(x_3) \geqq 3$. The construction proceeds inductively.

The proof of the next result is very similar to that of [4, Theorem 2.1]. Suggestions of Y. Katznelson have greatly simplified the original argument.

THEOREM 3.2. *Let A be a Boolean ring, ρ a function from A to the non negative reals satisfying*

1) $x \geqq y \Rightarrow \rho(x-y) \geqq |\rho(x)-\rho(y)|$,

2) $\rho(xy) \leqq \rho(x)\rho(y)$.

If A satises property (I), *then* $\sup_{y \in A} \rho(y) < \infty$.

Proof. Suppose $\sup_{y \in A} \rho(y) = \infty$. Then by Lemma 3.1 there exists $\{x_n\} \subset A$, $n = 1, 2, \cdots$, $x_n x_m = 0$, $n \neq m$, for which $\lim_{n \to \infty} \rho(x_n) = \infty$. Choose distinct elements y_{jk}, $j, k = 1, 2, \cdots$, from the sequence $\{x_n\}$ such that

$$\rho(y_{jk}) > 2^{j+k}.$$

Since A satisfies (I), by induction we may define an orthogonal sequence $\{y_j\} \subset A$ such that $y_j y_{jk} = y_{jk}$, $y_i y_{jk} = 0$, $i \neq j$. For each integer j select an integer k_j large enough so that $2^{k_j} > \rho(y_j)$. The sequences $\{y_{jk_j}\}$, $\{y_j - y_{jk_j}\}$ satisfy the hypothesis of property (I). Therefore there exists $y \in A$ satisfying $y y_{jk_j} = y_{jk_j}$, $y(y_j - y_{jk_j}) = 0$. Therefore $y y_j = y y_{jk_j} = y_{jk_j}$. Applying (2) we have

$$2^{j+k_j} < \rho(y_{jk_j}) \leqq \rho(y)\rho(y_j) < \rho(y) 2^{k_j}.$$

Therefore $\rho(y) > 2^j$ for each integer j, which is impossible.

The application of these results to Banach algebras with commuting idempotents is straight forward. For $p, q \in P_{\mathfrak{A}}$ we define $p \oplus q = p + q - 2pq$. Then $P_{\mathfrak{A}}$ is a Boolean ring with respect to this 'symmetric difference' addition and ordinary multiplication. Since for orthogonal idempotents $p \oplus q = p + q$, the norm satisfies condition (1), and we have the following result.

COROLLARY 3.3. *Let $\mathfrak{A}$ be a Banach algebra such that the elements of $P_{\mathfrak{A}}$ commute. Then if $P_{\mathfrak{A}}$ satisfies property* (I), *$P_{\mathfrak{A}}$ is a bounded set.*

If now $\mathfrak{A}$ is commutative and $\Phi_{\mathfrak{A}}$ is totally disconnected, then property (I) for $P_{\mathfrak{A}}$ can be characterized topologically in $\Phi_{\mathfrak{A}}$. In fact, in light of Lemma 2.1 and the theorem of Silov, $P_{\mathfrak{A}}$ satisfies (I) if and only if $P_{C(\Phi_{\mathfrak{A}})}$ satisfies (I). However, it is easily verified [10, p. 1619] that the latter condition holds if and only if disjoint open F_σ sets in $\Phi_{\mathfrak{A}}$ have disjoint closures. Such spaces are called F spaces by Gillman and Henriksen and are extensively studied in [7]. Examples include the Stone spaces of σ-complete Boolean algebras, the Čech compactification of the integers $\beta(N)$, and also $\beta(N) - N$. The latter is not the Stone space of a σ-complete Boolean algebra as is remarked in [10].

Combining these remarks with Theorem 2.4 and Corollary 3.3 we have

THEOREM 3.4. *If $\mathfrak{A}$ is a commutative Banach algebra and $\Phi_{\mathfrak{A}}$ is a totally*

disconnected F space, then $\mathfrak{A}$ *is strongly decomposable and* $\mathfrak{A}/\mathfrak{R} \simeq C(\Phi_{\mathfrak{A}})$. *The strong decomposition is in addition unique.*

COROLLARY 3.5. *If* Ω *is a totally disconnected compact F space, then* $C(\Omega)$ *is the only semi-simple commutative Banach algebra with* Ω *as maximal ideal space.*

It is an open question whether the class of totally disconnected compact F spaces is characterized by the property of Corollary 3.5.

IV. In this section we shall consider commutative algebras $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ for a compact space Ω, and give two conditions on the radical which will imply decomposability of $\mathfrak{A}$ for appropriate spaces Ω.

We note first that if σ is an isomorphism of $\mathfrak{A}/\mathfrak{R}$ onto $C(\Omega)$ then Ω is homeomorphic with $\Phi_{\mathfrak{A}}$. This follows easily since σ^{-1} followed by the Gelfand isomorphism of $\mathfrak{A}/\mathfrak{R}$ is a continuous isomorphism μ of $C(\Omega)$ onto a separating subalgebra of $C(\Phi_{\mathfrak{A}})$. By the result of Rickart [14, Theorem 1] discussed in § II, μ maps onto $C(\Phi_{\mathfrak{A}})$, and Ω is therefore homeomorphic with $\Phi_{\mathfrak{A}}$ by the classical result of Stone.

It is shown in § V that $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ by itself is not enough to guarantee the decomposability of $\mathfrak{A}$. However, if $\mathfrak{A}$ is decomposable the following general result holds.

THEOREM 4.1. *Let* $\mathfrak{A}$ *be a commutative Banach algebra such that* $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$. *If* $\mathfrak{A}$ *is decomposable, it is strongly decomposable, and the strong decomposition is unique.*

Proof. Suppose $\mathfrak{A} = \mathfrak{B}' \oplus \mathfrak{R}$ and let σ be the induced isomorphism of $C(\Omega)$ onto $\mathfrak{B}'$. By [4, Theorem 2.3] there exists a continuous isomorphism μ of $C(\Omega)$ into $\mathfrak{A}$ such that $\sigma - \mu$ maps $C(\Omega)$ into $\mathrm{Rad}(\overline{\mathfrak{B}'})$. If $\mathfrak{B} = \mu(C(\Omega))$, then $\mathfrak{B}$ is closed and $\mathfrak{B} \cap \mathfrak{R} = \{0\}$. Now $\mathrm{Rad}(\overline{\mathfrak{B}'}) = \overline{\mathfrak{B}'} \cap \mathfrak{R}$. It follows easily that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$, so $\mathfrak{A}$ is strongly decomposable.

We next prove the uniqueness of the closed subalgebra $\mathfrak{B}$. For $a \in \mathfrak{A}$ let T_a be the rgeular representation of $\mathfrak{A}$ in the algebra of bounded operators on $\mathfrak{A}$. The corespondence $f \to T_{\mu(f)}{}^*$ is a bicontinuous isomorphism of $C(\Omega)$ into the algebra of bounded operators on the conjugate space $\mathfrak{A}^*$. Thus in the terminology of Dunford [5, Theorem 18], $T_b{}^*$ for $b \in B$, is a scalar type spectral operator of class $\mathfrak{A}$ in $\mathfrak{A}^*$, and if $a \in \mathfrak{A}$, $T_a{}^* = T_b{}^* + T_r{}^*$ is a spectral operator in $\mathfrak{A}^*$ of class $\mathfrak{A}$ since $T_r{}^*$ is a commuting generalized nilpotent. By an important theorem of Dunford [5, Theorem 8] the splitting of a spectral operator into scalar and commuting generalized nilpotent parts is unique.

Thus if $\mathfrak{A} = \mathfrak{B}_1 \oplus \mathfrak{R}$, where $\mathfrak{B}_1$ is a closed subalgebra different from $\mathfrak{B}$, there will be an element a in $\mathfrak{A}$ for which the spectral operator T_a* has two distinct splittings. This contradiction completees the proof.

THEOREM 4.2. *Let $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ and assume Ω is totally disconnected. If there is an integer n such that $x^n = 0$,[8] $x \in \mathfrak{R}$, then $P_{\mathfrak{A}}$ is a bounded set. $\mathfrak{A}$ is therefore strongly decomposable and the decomposition is unique.*

Proof. Let $\sigma: \mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ and let ν be the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{R}$. Since $\mathfrak{A}/\mathfrak{R}$ is complete under the norm $\| \nu(x) \| = \inf_{r \in \mathfrak{R}} \| x + r \|$, σ is a homeomorphism. Therefore there is a constant $\alpha > 0$ such that for each $x \in \mathfrak{A}$ we can choose a corresponding $r \in \mathfrak{R}$ for which $\| x + r \| \leqq \alpha \| (\sigma\nu)(x) \|$. In particular, if $p \in P_{\mathfrak{A}}$, there is an element $r \in \mathfrak{R}$ such that $\| p + r \| \leqq \alpha$. Next we note that since $r^n = 0$, $r \in \mathfrak{R}$,

$$(p+r)^{n+k} = \sum_{j=0}^{n-1} C_j^{n+k} p r^j, \qquad k = 0, 1, 2, \cdots.$$

We assert next that there exist scalars $\lambda_0, \cdots, \lambda_{n-1}$ independent of p and r such that

$$p = \lambda_0 (p+r)^n + \lambda_1 (p+r)^{n+1} + \cdots + \lambda_{n-1} (p+r)^{2n-1}.$$

Expanding the right hand side of this expression and collecting terms of like powers of r this just means that $\lambda_0, \cdots, \lambda_{n-1}$ must be solutions of the equation

$$z_0 + z_1 + \cdots + z_{n-1} = 1$$
$$C_k^n z_0 + C_k^{n+1} z_1 + \cdots + C_k^{2n-1} z_{n-1} = 0, \qquad k = 1, 2, \cdots, n-1.$$

A solution certainly exists since it may be easily verified that the determinant of coefficients is non zero. Next for $p \in P_{\mathfrak{A}}$ choose $r \in \mathfrak{R}$ such that $\| p + r \| \leqq \alpha$. Then if $\lambda_0, \cdots, \lambda_{n-1}$ are solution of the above equations, we have

$$\begin{aligned} \| p \| &= \| \lambda_0 (p+r)^n + \cdots + \lambda_{n-1} (p+r)^{2n-1} \| \\ &\leqq | \lambda_0 | \, \| p + r \|^n + \cdots + | \lambda_{n-1} | \, \| p + r \|^{2n-1} \\ &\leqq \alpha^n \sum_{k=0}^{n-1} | \lambda_k | \, \alpha^k. \end{aligned}$$

Since this bound is independent of p, $P_{\mathfrak{A}}$ is a bounded set. An application of Theorem 2.4 completes the proof. This argument yields the following slightly more general result.

COROLLARY 4.3. *Let $\mathfrak{A}$ be a commutative Banach algebra with nilpotent*

[8] An equivalent hypothesis is that $\mathfrak{R}$ be nilpotent. This foolows easily from Newton's identities for the symmetric functions.

radical. Let S be a collection of idempotents of $\mathfrak{A}$, and $\nu(S)$ the natural image in $\mathfrak{A}/\mathfrak{R}$. If $\nu(S)$ is bounded in $\mathfrak{A}/\mathfrak{R}$, then S is bounded in $\mathfrak{A}$.

If few idempotents are present in the algebra $\mathfrak{A}$, the tools for proving $\mathfrak{A}$ decomposable are very meagre. However, if $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ and $\dim \mathfrak{R} < \infty$, then a device due to Arens [1, 2] enables us to reduce the problem to previous results. Specifically, if $\mathfrak{A}$ is a Banach algebra and $\mathfrak{A}^{**}$ is its second conjugate space, Arens defined a multiplication in $\mathfrak{A}^{**}$ under which $\mathfrak{A}^{**}$ is a Banach algebra containing $\mathfrak{A}$ as a closed weak * dense subalgebra under the natural imbedding. The reader should consult [2] for details of this construction. We note one fact which is important in the following. If $x \in \mathfrak{A}^{**}$ and $\{x_\alpha\} \subset \mathfrak{A}$ is a directed family converging to x in the weak * topology, then for each $y \in \mathfrak{A}$ and each $z \in \mathfrak{A}^{**}$, $yx_\alpha \to yx$ and $x_\alpha z \to xz$ in the weak * topology [2, Theorem 3.2]. This clearly shows that if $\mathfrak{A}$ is commutative, $\mathfrak{A}$ is contained in the center of $\mathfrak{A}^{**}$.

In the following discussion we will always consider $\mathfrak{A}^{**}$ as a Banach algebra without making specific mention of the fact. For a continuous linear map ν, we let ν^{**} denote the canonical second adjoint map, and for $\mathfrak{K} \subset \mathfrak{A}$ we let

$$\mathfrak{K}^{\perp} = \{f \in \mathfrak{A}^* : f(\mathfrak{K}) = 0\}.$$

LEMMA 4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and ν be a continuous homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then ν^{**} is a homomorphism of $\mathfrak{A}^{**}$ into $\mathfrak{B}^{**}$ (with respect to the Arens multiplication). If ν maps $\mathfrak{A}$ onto $\mathfrak{B}$ with kernel $\mathfrak{K}$, then ν^{**} maps $\mathfrak{A}^{**}$ onto $\mathfrak{B}^{**}$ with kernel $\mathfrak{K}^{\perp\perp}$.*

Proof. Note first that ν^{**} is continuous with respect to the weak * topologies in $\mathfrak{A}^{**}$ and $\mathfrak{B}^{**}$. Since $\mathfrak{A}$ is weak * dense in $\mathfrak{A}^{**}$, for $x, y \in A^{**}$ pick nets $\{x_\alpha\}$, $\{y_\beta\}$ in $\mathfrak{A}$ converging, weak *, to x, y respectively. By the appropriate weak * continuity of multiplication in $\mathfrak{A}^{**}$ and $\mathfrak{B}^{**}$ we have for fixed α, $\nu^{**}(x_\alpha)\nu^{**}(y_\beta) \to \nu^{**}(x_\alpha)\nu^{**}(y)$, and $\nu^{**}(x_\alpha y_\beta) \to \nu^{**}(x_\alpha y)$. Since ν^{**} agrees with ν on $\mathfrak{A}$, it follows that $\nu^{**}(x_\alpha y) = {}_\alpha{}^{**}(x_\alpha)\nu^{**}(y)$. Applying this argument again we have $\nu^{**}(xy) = \nu^{**}(x)\nu^{**}(y)$. If ν maps $\mathfrak{A}$ onto $\mathfrak{B}$ with kernel $\mathfrak{K}$, then ν^* is one to one and $\nu^*(\mathfrak{B}^*) = \mathfrak{K}^{\perp\perp}$. The last two statements now follows easily.

THEOREM 4.4. *Let $\mathfrak{A}$ be a commutative Banach algebra with finite dimensional radical $\mathfrak{R}$. If $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ for a compact Hausdorff space Ω, then $\mathfrak{A}$ is strongly decomposable and the decomposition is unique.*

Furthermore, if $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$, the annihilator of $\mathfrak{R}$ in $\mathfrak{B}$ has deficiency in $\mathfrak{B} \leqq \dim \mathfrak{R}$.

Proof. We note first that $C^{**}(\Omega)$ is isometrically isomorphic to $C(\Omega')$ for some extremely disconnected compact Hausdorff space Ω'. This is a consequence of results of Arens and Grothendieck. In [1] Arens showed that $C^{**}(\Omega)$ is isometrically isomorphic to $C(\Omega')$ for a compact space Ω'. But by [9, Theorem 2] if $C(\Omega')$ is an adjoint space, Ω' must be extremely disconnected.

If ν is the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{R}$, $\sigma: \mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$, then $\mu = \sigma\nu$ is a continuous homomorphism of $\mathfrak{A}$ onto $C(\Omega)$ with kernel $\mathfrak{R}$. By the previous lemma μ^{**} is a homomorphism of $\mathfrak{A}$ onto $C^{**}(\Omega)$ with kernel $\mathfrak{R}^{\perp\perp}$ and since $\dim \mathfrak{R} < \infty$, $\mathfrak{R} = \mathfrak{R}^{\perp\perp}$. We assert next that $\mathfrak{R}$ is the radical of $\mathfrak{A}^{**}$.

Let $\mathfrak{N}$ be the radical of $\mathfrak{A}^{**}$. If $x \in \mathfrak{R}$ $\| x^n \|^{1/n} \to 0$, and therefore $\| \nu^{**}(x)^n \|^{1/n} \to 0$ implying $\nu^{**}(x) = 0$ or $x \in \mathfrak{R}$. Now the radical of $\mathfrak{A}^{**}$ contains all topologically nilpotent ideals of $\mathfrak{A}^{**}$, if $x \in \mathfrak{R}$, $y \in \mathfrak{A}^{**}$, then

$$\| (xy) \|^{1/n} = \| x^n y^n \|^{1/n} \leqq \| y \| \cdot \| x^n \|^{1/n} \to 0.$$

Therefore the ideal $x\mathfrak{A}^{**}$ is topologically nilpotent, and consequently $x \in \mathfrak{N}$.

Since R is contained in the center of $\mathfrak{A}^{**}$, the idempotents of $\mathfrak{A}^{**}$ commute by Lemma 2.3. Applying Lemma 2.1 it is easy to see that since $P_{C(\Omega')}$ satisfies property (I), $P_{\mathfrak{A}^{**}}$ does also. Therefore, it follows by Corollary 3.3 that $P_{\mathfrak{A}^{**}}$ is a bounded set, and consequently by Theorem 2.4 there exists a unique subalgebra $\mathfrak{B}'$ of $\mathfrak{A}^{**}$, necessarily closed, such that $\mathfrak{A}^{**} = \mathfrak{B}' \oplus \mathfrak{N}$. If $\mathfrak{B} = \mathfrak{B}' \cap \mathfrak{A}$, $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ since $\mathfrak{R} \subset \mathfrak{A}$. The uniqueness of $\mathfrak{B}$ in $\mathfrak{A}$ is a consequence of Theorem 4.1.

To prove the last statement we adapt a technique of Feldman [6, Theorem 2]. Let $\mathfrak{J} = \{x \in \mathfrak{B} \mid x\mathfrak{R} = 0\}$. $\mathfrak{J}$ is a closed ideal in $\mathfrak{B}$. Therefore $\mu(\mathfrak{J})$ is a closed ideal in $C(\Omega)$, and consequently there exists a closed set F of Ω such that $\mu(\mathfrak{J})$ is the set of all functions in $C(\Omega)$ which vanish on F. We assert F is finite with cardinality $\leqq n$, the dimension of $\mathfrak{R}$. If the cardinality of F exceeds n, there exist elements $x_1, \cdots, x_{n+1}$, $y_1, \cdots, y_{n+1} \in \mathfrak{B}$, $y_i x_i = x_i$, $i = 1, \cdots, n+1$; $y_i y_j = 0$, $i \neq j$; and $x_i \mathfrak{R} \neq \{0\}$, $i = 1, \cdots, n+1$. Therefore, for each index i pick $r_i \in R$ such that $x_i r_i \neq 1$. Since $\dim R = n$, $x_1 r_1$, $\cdots, x_{n+1} r_{n+1}$ are linearly dependent. Therefore for some i

$$x_i r_i = \sum_{\substack{j=1 \\ i \neq j}}^{n+1} \lambda_j x_j r_j.$$

But then

$$x_i r_i = y_i x_i r_i = \sum_{\substack{j=1 \\ i \neq j}}^{n+1} \lambda_j y_i x_j r_j = 0$$

which is a contradiction.

Clearly the deficiency of $\mathfrak{J}$ equals the cardinality of F since F is a finite set. This completes the proof.

V. By Theorem 4.2 if $\mathfrak{A}/\mathfrak{R} = C(\Omega)$, Ω is totally disconnected, and $\mathfrak{R}^n = 0$, then $\mathfrak{A}$ is (strongly) decomposable. In this section we shall show that this theorem is false if either of these hypotheses are dropped. We first show that $\mathfrak{A}/\mathfrak{R} \simeq C(\Omega)$ is not enough to guarantee the decomposability of $\mathfrak{A}$. Our construction depends on the following result.

LEMMA 5.1. *There exists a commutative n-dimensional Banach algebra* $\mathfrak{A}_n$ *with radical* $\mathfrak{R}_n$ *having the following properties:*

(1) $\mathfrak{A}_n = \{\lambda p_n\} \oplus \mathfrak{R}_n$, $p_n{}^2 = p_n$,

(2) $\| p_n \| = 2^n$, $\inf_{r \in \mathfrak{R}_n} \| p_n + r \| = 2$.

Proof. Let $\{e_0, \cdots, e_{n-1}\}$ be a basis for an n-dimensional vector space $\mathfrak{A}_n$ over the complex field. For multiplication we define $e_0{}^2 = e_0$, $e_0 e_j = e_j e_0 = 0$, $j \neq 0$. For $j, k \geqq 1$,

$$e_j e_k = e_{j+k} \quad \text{if} \quad j + k \leqq n - 1$$
$$= 0 \quad \text{otherwise.}$$

Clearly $\mathfrak{A}_n$ is a commutative associative algebra, and if $p_n = e_0$, (1) follows. Now let $r = e_1$, then $r^k = e_k$, $1 \leqq k < n$, and $r^k = 0$, $k \geqq n$. Clearly $r, r^2, \cdots, r^{n-1}$ span $\mathfrak{R}_n$. If we let $f = p_n/2 + r$, then for $k < n$, $f^k = p_n/2^k + r^k$, and for $k > n$, $f^k = p_n/2^k$. Clearly $f, \cdots, f^k$ are independent and span $\mathfrak{A}_n$. Therefore each x in $\mathfrak{A}_n$ may be written $x = \sum_{k=1}^{n} \alpha_k f^k$, and we define $\| x \| = \sum_{k=1}^{n} | \alpha_k |$. Under this norm $\mathfrak{A}_n$ is obviously a Banach space.

To show $\| xy \| \leqq \| x \| \| y \|$ for $x, y \in \mathfrak{A}_n$ observe first that $f^k f^j = f^{k+j}$ if $k + j \leqq n$, and $f^k f^j = 2^{-(k+j-n)} f^n$ if $k + j > n$. Therefore if

$$y = \alpha_1 f + \cdots + \alpha_n f^n,$$

$$f^k y = \alpha_1 f^{k+1} + \cdots + \alpha_{n-k-1} f^{n-1} + (\alpha_{n-k} + 2^{-1}\alpha_{n-k-1} + \cdots + 2^{-k}\alpha_n) f^n.$$

Therefore

$$\| f^k y \| = | \alpha_1 | + \cdots + | \alpha_{n-k-1} | + | \alpha_{n-k} + 2^{-1}\alpha_{n-k+1} + \cdots + 2^{-k}\alpha_n | \leqq | \alpha_1 | + \cdots + | \alpha_n | = \| y \|.$$

Hence if $x = \beta_1 f + \cdots + \beta_n f^n$, $\| xy \| < | \beta_1 | \| fy \| + \cdots + | \beta_n | \| f^n y \| \leqq \| x \| \| y \|$. Lastly, since $f^n = 2^{-n} p_n$ and $\| f \| = 1$ we have $\| p_n \| = 2^n$. Also, since $\| f \| = \| p_n/2 + r \| = 1$, $\inf_{r \in \mathfrak{R}_n} \| p_n + r \| \leqq 2$. But if there exists $t \in \mathfrak{R}_n$ such that $\| p_n + t \| < 2$, then $\| p_n/2 + t/2 \| < 1$ and $\| (p_n/2 + t/2)^n \| < 1$. However, $p_n t = 0$ for $t \in \mathfrak{R}$. Therefore $(p_n/2 + t/2)^n = p_n/2^n$, and $\| p_n \| < 2^n$. This is a contradiction.

We now define $\mathfrak{A}_0 \equiv \{\{x_n\} : x_n \in \mathfrak{A}_n, \lim_{n\to\infty} \| x_n \| = 0\}$. If the algebraic operations are defined coordinate-wise and for $x \in A_0$, $\| x \| = \sup_n \| x_n \|$, then $\mathfrak{A}_0$ is a commutative Banach algebra. Let $\mathfrak{A}$ be the algebra obtained by adjoining an identity e to $\mathfrak{A}_0$ in the usual way. For $y = \alpha e + x$, $x \in \mathfrak{A}_0$, $\| y \| = | \alpha | + \| x \|$. We note that if q is an idempotent of $\mathfrak{A}$, q or $e - q$ belongs to $\mathfrak{A}_0$, since for any idempotent in $\mathfrak{A}_0$ all but a finite number of coordinates are zero. If $x \in \mathfrak{A}_n$, then the natural isometric image of x in $\mathfrak{A}$ will be denoted by x_n. Since $\| p_n \| = 2^n$, and $\| q \| \geqq \| p \|$ if $qp = p$, it is clear that any uniformly bounded family of orthogonal idempotents in $\mathfrak{A}$ is finite.

THEOREM 5.2. *If $\mathfrak{R}$ is the radical of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{R} \cong (c)$, the ring of convergent sequences, and $\mathfrak{A}$ is not decomposable.*

Proof. We verify first that $\Phi_{\mathfrak{A}_0}$ is totally disconnected. If $\mathfrak{A}' = \{x \in \mathfrak{A} : x_n \equiv 0, n \text{ sufficiently large}\}$, then $\mathfrak{A}'$ is dense in $\mathfrak{A}_0$, and any $x \in \mathfrak{A}'$ may be written $x = \sum_{k=1}^{n} \lambda_k p_k + r$, $r \in \mathfrak{R}$. If ϕ, ϕ' are two distinct points in $\Phi_{\mathfrak{A}_0}$, there exists an $x \in \mathfrak{A}'$ for which $\phi(x) \neq \phi'(x)$. In fact for some k, $\phi(p_k) \neq \phi'(p_k)$, which shows that $\Phi_{\mathfrak{A}_0}$ is totally disconnected.

For $x \in \mathfrak{A}$ let $\bar{x}$ be the corresponding element in $\mathfrak{A}/\mathfrak{R}$. If $\bar{p}$ is any idempotent of $\mathfrak{A}/\mathfrak{R}$ then $\| \bar{p} \| \leqq 3$. For by Lemma 2.1 there is a unique idempotent $q \in \mathfrak{A}$ such that $\bar{q} = \bar{p}$. Hence, assuming $q \neq 0$, if $q \in \mathfrak{A}_0$, $\| \bar{p} \| = \inf_{r \in R} \| q + r \| = 2$ if $e - q \in \mathfrak{A}_0$, $\| e - \bar{p} \| = 2$ and $\| \bar{p} \| \leqq 3$. An application of Theorem 2.4 shows that the span of the idempotents of $\mathfrak{A}/\mathfrak{R}$ is dense in $\mathfrak{A}/\mathfrak{R}$. It is now easily verified that for $\xi = \{\xi_k\} \in (c)$, if $\xi_0 = \lim_{k\to\infty} \xi_k$ then the mapping $\xi \to \xi_0 \bar{e} + \sum_{k=1}^{\infty} (\xi_k - \xi_0) \bar{p}_k$ is a bicontinuous isomorphism of (c) onto $\mathfrak{A}/\mathfrak{R}$.

If $\mathfrak{A}$ were decomposable, then there would exist an isomorphism ν of (c) into $\mathfrak{A}$. But if $\{e_i\}$ is any infinite collection of orthogonal idempotents of (c), by [4, Corollary 2.2] $\{\nu(e_i)\}$ is a bounded set in $\mathfrak{A}$. Any such family is necessarily finite, hence no such isomorphism can exist. $\mathfrak{A}$ is therefore not decomposable.

Actually, the following stronger result is true. If σ is any homomorphism of (c) into $\mathfrak{A}$, then σ has finite dimensional range. A sketch of the proof is as follows. By [4, Theorem 4.3] we may write $\sigma = \mu + \lambda$, where μ is continuous and λ maps (c) into the radical of $\mathfrak{A}$. The kernel $\mathfrak{K}$ of μ is the set of convergent sequences vanishing on some closed set F of $\Phi_{(c)}$. The latter is of course just the one point compactification of the positive integers.

Using the continuity of μ and the fact that any bounded family of orthogonal idempotents in $\mathfrak{A}$ is finite, it is easy to see that $\mu(z) = 0$ if infinitely many of the coordinates of z are different from zero. Therefore, F is a finite set of points, and since $(c)/\mathfrak{R} \simeq C(F)$, μ has finite dimensional range. To show $\lambda \equiv 0$, one observes first that λ is a homomorphism on (c_0). Since for any $z \in (c_0)$, $[\lambda(z)^n] = 0$, we have for $z \geqq 0$, $z \in (c_0)$, $\lambda(z)_n = [\lambda(z^{1/n})^n]_n = 0$. Consequently, $\lambda \equiv 0$.

Next we show that $\mathfrak{R}^n = 0$ is also not enough to guarantee decomposability. The example is similar to the previous one.

LEMMA 5.3. *For each integer n there exists a three dimensional commutative Banach algebra $\mathfrak{A}_n$ with two dimensional radical $\mathfrak{R}_n$ having the following properties:*

(1) $\mathfrak{A}_n = \{\lambda p_n\} \oplus \mathfrak{R}_n$, $p_n^2 = p_n$,

(2) $\| p_n \| = n^3$; $\inf_{r \in \mathfrak{R}_n} \| p_n + r \| = n$.

Proof. Let $\mathfrak{A}_n$ be the three dimensional commutative algebra generated by elements p_n, r, s, where $p_n^2 = p_n$; $p_n r = r p_n = 0$; $r^2 = s$ and $rs = 0$. Now elements $f_1 = p/n + r$, $f_2 = p/n^2 + s$, $f_3 = p/r^3$ also form a basis for $\mathfrak{A}_n$. Therefore, if $x \in \mathfrak{A}_n$ and $x = \sum_{i=1}^{3} \alpha_i f_i$, we define $\| x \| = \sum_{i=1}^{3} | \alpha_i |$. We leave to the reader the verification that $\mathfrak{A}_n$ is a Banach algebra under this norm. Assertaions (1) and (2) are clear.

We now define $\mathfrak{A} = l_1(\mathfrak{A}_n) = \{\{x_n\} : x_n \in \mathfrak{A}_n, \sum_{n=1}^{\infty} \| x_n \| < \infty\}$. The algebraic operations are defined pointwise and for $x \in \mathfrak{A}$, $\| x \| = \sum_{n=1}^{\infty} \| x_n \|$.

THEOREM 5.4. *If $\mathfrak{R}$ is the radical of the above algebra $\mathfrak{A}$, $\mathfrak{R}^3 = \{0\}$, and $\mathfrak{A}$ is not decomposable.*

Proof. Clearly $\mathfrak{R} = \{x \in \mathfrak{A} : x_n \in \mathfrak{R}_n\}$, and since $\mathfrak{R}_n^3 = \{0\}$, it follows that $\mathfrak{R}^3 = \{0\}$. Let $\bar{p}_n \in \mathfrak{A}/\mathfrak{R}$ be the images of the primitive idempotents p_n. By the construction $\| \bar{p}_n \| = \inf_{r \in \mathfrak{R}} \| p_n + r \| = \inf_{r \in \mathfrak{R}_n} \| p_n + r \| = n$. If there exists an isomorphism μ of $\mathfrak{A}/\mathfrak{R}$ into $\mathfrak{A}$, then by [4, Corollary 2.2] there exists a constant M such that

$$\| \mu(\bar{p}_n) \| \leqq M \| \bar{p}_n \|^2 = Mn^2, \qquad n = 1, 2, \cdots.$$

But by the construction it is clear that for each n

$$\sup_{k=1,\cdots,n} \| \mu(\bar{p}_k) \| \geqq n^3.$$

These two facts are obviously incomparable, hence the theorem follows.

In the above example, $\mathfrak{A}/\mathfrak{R}$ is isometrically isomorphic with the algebra $\mathfrak{B}$ of sequences $\{\alpha_n\}$ such that $\sum_{n=1}^{\infty} n \mid \alpha_n \mid < \infty$. For if $\bar{x} = \sum_{n=1}^{N} \alpha_n \bar{p}_n$, then

$$\| \bar{x} \| = \inf_{r \in \mathfrak{R}} \| \sum_{n=1}^{N} \alpha_n p_n + r \| = \inf_{r \in \mathfrak{R}} \sum_{n=1}^{N} \| \lambda_n p_n + r_n \|$$

$$= \sum_{n=1}^{N} \inf_{r_n \in \mathfrak{R}_n} \| \alpha_n p_n + r_n \| = \sum_{n=1}^{N} n \mid \alpha_n \mid .$$

Thus we have an alegbraic isometry of a dense subalgebra of $\mathfrak{B}$ onto a dense subalgebra of $\mathfrak{A}/\mathfrak{R}$ which can be extended to an isometry of $\mathfrak{B}$ onto $\mathfrak{A}/\mathfrak{R}$. The following result shows that for algebras for which $\mathfrak{A}/\mathfrak{R} \simeq l_1$ and $\mathfrak{R}^n = \{0\}$ no such construction as the above is possible. We are assuming coordinate-wise multiplication in l_1.

THEOREM 5.5. *Let A be a commutative Banach algebra such that $A/R \simeq l_1$. Then if $\mathfrak{R}^n = \{0\}$ for some integer n, $\mathfrak{A}$ is strongly decomposable, and the strong decomposition is unique.*

In the light of Corollary 4.3 this will follow from the following:

THEOREM 5.6. *Let $\mathfrak{A}/\mathfrak{R} \simeq l_1$. Then $\mathfrak{A}$ is decomposable if and only if the primitive idempotents of $\mathfrak{A}$ form a bounded set. If the primitive idempotents of $\mathfrak{A}$ are bounded, then $\mathfrak{A}$ is strongly decomposable, and the strong decomposition is unique.*

Proof. Suppose $\mathfrak{A}$ is decomposable. Then there exists an isomorphism μ of l_1 into $\mathfrak{A}$ which carries the set S of primitive idempotents of l_1 onto the set T of primitive idempotents of $\mathfrak{A}$. Since S is a bounded set in l_1, by [4, Corollary 2.2] $\mu(S) = T$ is a bounded set in $\mathfrak{A}$. Conversely, suppose T is a bounded set in $\mathfrak{A}$. Let ν be the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{R}$, λ the isomorphism of $\mathfrak{A}/\mathfrak{R}$ onto l_1, and let $\mu = \lambda\nu$. For each primitive idempotent $e_i \in l_1$, there exists a unique primitive idempotent $f_i \in \mathfrak{A}$ such that $\mu(f_i) = e_i$. Let $\mathfrak{B} = \{x \in \mathfrak{A} \mid x = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k f_k$, where $\sum_{k=1}^{\infty} \mid \alpha_k \mid < \infty\}$. By the boundedness of $\{f_k\}$, $\sum_{k=1}^{\infty} \alpha_k f_k$ converges absolutely. Therefore since μ is continuous, $\mu(x) = \sum_{k=1}^{\infty} \alpha_k e_k$, and μ maps $\mathfrak{B}$ in a one to one fashion onto l_1. For $x \in \mathfrak{B}$, $\| x \| \leqq \sum_{k=1}^{\infty} \mid \alpha_k \mid \| f_k \| \leqq M \| \mu(x) \|$; therefore $\mathfrak{B}$ is closed, and $\mathfrak{A}$ is strongly decomposable. The uniqueness of this strong decomposition can now be verified easily by adapting the method of Theorem 2.4.

We note that under the hypothesis of Theorem 5.5 the decomposition

will not in general be unique. To see this we define $\mathfrak{A} = l_1 \oplus \{\alpha r\}$, where $r^2 = r(l_1) = 0$. Let μ be the natural isometric injection of l_1 into $\mathfrak{A}$. If f is a discontinuous linear functional on l_1 which vanishes on $(l_1)^2$, then $\lambda(z) = \mu(z) + f(z)r$, for $z \in l_1$ defines an isomorphism of l_1 into $\mathfrak{A}$. If $\mathfrak{B}' = \lambda(l_1)$, then it is clear that $\mathfrak{A} = \mathfrak{B}' \oplus \mathfrak{R}$, and $\mathfrak{B}'$ is not closed in $\mathfrak{A}$.

UNIVERSITY OF CALIFORNIA, BERKELEY,
UNIVERSITY OF CALIFORNIA, LOS ANGELES.

REFERENCES.

[1] R. Arens, "Operations induced in functions classes," *Monatshefte für Mathematik*, vol. 55 (1951), pp. 1-19.

[2] ———, "The adjoint of a bilinear operation," *Proceedings of the American Mathematical Society*, vol. 2 (1951), pp. 839-848.

[3] W. G. Bade, "On Boolean algebras of projections and algebras of operators," *Transactions of the American Mathematical Society*, vol. 80 (1955), pp. 345-360.

[4] ——— and P. C. Curtis, Jr., "Homomorphisms of commutative Banach algebras," *American Journal of Mathematics*, vol. 82 (1960), pp. 589-608.

[5] N. Dunford, "Spectral operators," *Pacific Journal of Mathematics*, vol. 4 (1954), pp. 321-354.

[6] C. Feldman, "The Wedderburn principal theorem in Banach algebras," *Proceedings of the American Mathematical Society*, vol. 2 (1951), pp. 771-777.

[7] L. Gillman and M. Henriksen, "Rings of continuous functions in which every finitely generated ideal is principal," *Transactions of the American Mathematical Society*, vol. 82 (1956), pp. 366-391.

[8] G. Glaeser, "Étude de certaines algèbres tay loriennes," *Journal d'Analyse Mathematique*, vol. 6 (1958), pp. 1-124.

[9] A. Grothendieck, "Une caractérisation des espaces *L'*," *Canadian Journal of Mathematics*, vol. 7 (1955), pp. 552-561.

[10] M. Henriksen, "Some remarks on a paper of Aronszajn and Panitchpakdi," *Pacific Journal of Mathematics*, vol. 7 (1957), pp. 1619-1621.

[11] I. Kaplansky, "Normed algebras," *Duke Mathematical Journal*, vol. 16 (1949), pp. 399-418.

[12] H. A. Kober, "A theorem on Banach spaces," *Compositio Mathematica*, vol. 7 (1939), pp. 135-140.

[13] L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953.

[14] C. E. Rickart, "On spectral permanence for certain Banach algebras," *Proceedings of the American Mathematical Society*, vol. 4 (1953), pp. 191-196.

[15] ———, *General theory of Banach algebras*, New York, 1960.

[16] G. E. Silov, "On a decomposition of a commutative normed ring in a direct sum of ideals," *Mathematičeskiĭ Sbornik N. S.*, vol. 32 (1954), pp. 353-364.

[17] D. Zelinsky, "Raising idempotents," *Duke Mathematical Journal*, vol. 21 (1954), pp. 315-322.

FUNCTIONS THAT ARE HARMONIC OR ZERO.*[1]

By JOHN W. GREEN.

1. Introduction. In a paper on integral characterizations of harmonic functions, E. F. Beckenbach [1] considered the following proposition:

THEOREM 1. *Let $u(x,y)$ be of class C' in a domain D and have the property that at each point (x_0,y_0) of D either $u(x_0,y_0)=0$ or u is harmonic in some neighborhood of (x_0,y_0). Then u is harmonic in D.*

If u is only of class C^0, the theorem is false, as the example $u=|x|$ shows. If u is of class C'', the result is trivial, since it follows immediately that $\nabla^2 u \equiv 0$. In [1] is given a proof of the theorem under the additional hypothesis that u_{xx} and u_{yy} are summable.[2] In the present paper, Theorem 1 is proved without any addditional assumptions on u, applications to potential theory made, and an analogous theorem proved for analytic functions.

2. A preliminary lemma. Let E be the set of points of D at which u is not harmonic. The set E is closed relative to D and $u=0$ on E. We prove the following lemma:

LEMMA 1. $u_x = u_y = 0$ *on* E.

Let (x_0,y_0) be a point of E at which $u_y \neq 0$. Now E is contained in the locus of the equation $u(x,y)=0$; hence all points of E in some neighborhood of (x_0,y_0) will lie on the curve $y=f(x)$ of class C' which is the solution of the implicit equation $u(x,f(x))=0$. The function u is harmonic above and below this curve and continuous together with u_x and u_y across it. Conjugate functions $v+$ and $v-$ to u above and below the curve can be constructed by the usual integral formula $\int (\partial u/\partial n)\,ds$. Since integrals along the curves are allowed, $v+$ and $v-$ differ on the curve by a constant, which can be taken to be zero. By Painleve's theorem on the continuation of an analytic function

* Received March 7, 1960.

[1] The preparation of this paper was sponsored, in part, by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

[2] It is perhaps not completely clear in [1] that what is needed and used there is that u_x be absolutely continuous with respect to x for almost all y, that u_{xx} be summable over D, and the same holds with letters interchanged.

across a rectifiable curve on which it remains continuous, one sees that u continues harmonically into itself across the curve. Therefore (x_0, y_0) does not belong to E after all. The case of u_x is handled analogously.

3. A special case of Theorem 1. We now prove Theorem 1 in the case D is a circle and u is harmonic on the boundary of D; in this case $\bar{E} \subset D$.

LEMMA 2. *Let D' be that component of $D - E$ which extends to the boundary of D. Then u_{xx} and u_{yy} are summable over D'.*

Let w be u_x or u_y. In either case w is harmonic in D' and on the outer boundary of D'; it is continuous and equal to zero on the inner boundary, which is composed of points of E. It is clearly possible to define a function in D' with finite Dirichlet integral and with the same boundary values as w. Since the harmonic function with prescribed continuous boundary values, when it exists, minimizes the Dirichlet integral, w has finite Dirichlet integral over D'. Thus $|w_x|^2$ and $|w_y|^2$ are summable over D' and this implies that u_{xx} and u_{yy} are.

LEMMA 3. *Let f be a continuous function on (a, b) and F a closed set in (a, b). Let $f(x) = 0$ for x in F, and let f be absolutely continuous on each of the open intervals of which the set $(a, b) - F$ is formed. Finally let f' be summable on $(a, b) - F$. Then f is absolutely continuous on (a, b) and $f' = 0$ almost everywhere on F.*

Let x_0 be a point of F and set $g(x) = f'(x)$ for x in $(a, b) - F$ and $g(x) = 0$ for x in F. Define

$$h(x) = \int_{x_0}^{x} g(t)\,dt.$$

If $x \in F$, then $h(x) = \sum \int_{\alpha_i} f'(t)\,dt$, where α_i are the intervals of $(a, b) - F$ lying between x_0 and x. But if $\alpha_i = (a_i, b_i)$,

$$\int_{\alpha_i} f'(t)\,dt = f(b_i) - f(a_i) = 0.$$

Thus $h(x) = 0$ for x in F. If $x \in (a, b) - F$, let x_1 be one of the ends of the interval of $(a, b) - F$ in which x lies. Then

$$h(x) = \int_{x_0}^{x_1} g(t)\,dt + \int_{x_1}^{x} g(t)\,dt = 0 + \int_{x_1}^{x} f'(t)\,dt = f(x) - f(x_1) = f(x).$$

We see now that $f \equiv h$ and f is absolutely continuous. Since $f' = g$ almost everywhere, $f' = 0$ almost everywhere in F.

LEMMA 4. *For almost all y, $u_x(x,y)$ is absolutely continuous with respect to x, u_{xx} is summable over D, and $u_{xx}=0$ almost everywhere in $D-D'$.*

Let L_y represent the horizontal line with ordinate y. Since u_{xx} is summable over D', it is summable over $L_y \cdot D'$ for almost all y. For such a y we apply Lemma 3 to u_x, taking $(a,b)-F$ to be $L_y \cdot D'$. The set F consists of points of E, at which $u_x=0$, and of points of other components than D' of $D-E$. But such components would have as boundary points only points of E. Therefore $u \equiv 0$ in such components and $u_x=0$ on F. From Lemma 3 it follows that u_x is absolutely continuous in x, and $u_{xx}=0$ almost everywhere on F. By Fubini's theorem, $u_{xx}=0$ almost everywhere (two dimensional measure) on $D-D'$, and u_{xx} is summable over D.

The special case of Theorem 1 is at hand, since Beckenbach's result is applicable. However, for completeness, the following proof, slightly different from that in [1], is included.

It suffices to prove that $\int (\partial u/\partial n)\,ds=0$ around any rectangle in D with horizontal and vertical sides. For if this is so, one can define a conjugate function

$$v=\int_{(x_0,y_0)}^{(x,y)} (\partial u/\partial n)\,ds,$$

restricting the integration to a horizontal followed by a vertical segment, or vice-versa. Then u and v are easily seen to satisfy the Cauchy-Riemann equations.

Let then (x_0,y_0) and (x_1,y_1) be lower left and upper right vertices of a rectangle R in D. For almost all y

$$\int_{x_0}^{x_1} u_{xx}(x,y)\,dx = u_x(x_1,y)-u_x(x_0,y),$$

and so

$$\int\int_R u_{xx}\,dxdy=\int_{y_0}^{y_1} u_x(x_1,y)\,dy-\int_{y_0}^{y_1} u_x(x_0,y)\,dy.$$

If the corresponding equation with x and y interchanged is put down and added to this one, the right hand side of the resulting equation is the integral of $\partial u/\partial n$ around R. The left hand side is

$$\int\int_R (u_{xx}+u_{yy})\,dxdy=\int\int_{D'\cdot R}(u_{xx}+u_{yy})\,dxdy=0.$$

4. A lemma concerning analytic functions.

LEMMA 5. *Let K be a simply connected domain lying in the open unit*

circle D. Let f be analytic in K and continuous on $\bar{K}$. Let $f=0$ at all boundary points of K lying in D. Then if K has a boundary point in D, $f\equiv 0$ in K.

Let A be the set of boundary points of K at which $f=0$, and B the remaining boundary points of K, at which $f\neq 0$. By the continuity of f, B is a relatively open set on the unit circle $|z|=1$ and all points of B are accessible from K by straight line segments. Let

$$z=\phi(w)$$

map K conformally on the unit circle $|w|<1$. By known properties of the mapping function, B is carried into a relatively open set G on $|w|=1$ in a one-to-one fashion. If F is the complement of G on $|w|=1$, then A is carried into F. By this is meant that if a sequence w_i tends to a point of F, all accumulation points of $\phi(w_i)$ lie on A, and vice-versa.

Suppose K has a boundary point in D. Then F is of positive measure on $|w|=1$. This follows from a theorem of A. Ostrowski [2] which states that in the mapping of a simply connected domain on the unit circle, every neighborhood of any boundary point contains a set of accessible boundary points whose image on $|w|=1$ has positive measure.

Consider the function

$$g(w)=f(\phi(w)).$$

It is analytic and bounded in $|w|<1$, and tends to zero when w tends to a point of F. By a theorem of F. and M. Riesz (see for example [3], p. 156) $f\equiv 0$.

5. Proof of Theorem 1. We return to Theorem 1 in the general case. Let then u be of class C' in a domain D, and u be zero or harmonic at every point of D. Let P be a point of D at which u is not harmonic, and let D' be an open circle about P such that $\bar{D}'\subset D$. Let K be any connected component of $D'-E$.

We first note that K must be simply connected. If it were not, some simple closed curve lying in K would enclose points of E. By mapping the interior of this curve on the unit circle, we would have the situation covered in Section 3 and would conclude that there could be no points of E inside the curve after all.

Consider now the function

$$f(z)=u_x(x,y)-iu_y(x,y).$$

It is analytic in K and continuous in $\bar{K}$; furthermore it is zero at all boundary points of K lying in D', such boundary points being points of E. Also, since K is not all of D', there must be such boundary points. Therefore by Lemma 5, $f \equiv 0$ in K, and so $u_x \equiv u_y \equiv 0$ in K. Since $u = 0$ on the aforementioned boundary points of K, $u \equiv 0$ in K.

It follows that $u \equiv 0$ in $D' - E$, and since $u = 0$ on E, $u \equiv 0$ in D'. Thus u is harmonic in D' and P cannot be a point of E. This concludes the proof.

6. Applications. In [1] it is shown that if u is of class C' in D and if the Dirichlet integral of u over each circle of radius r differs from the integral of $u\partial u/\partial n$ around the perimeter by $o(r^2)$, then u is harmonic in D provided either (a) u does not vanish in D, or (b) u satisfies the conditions stated in the footnote to § 1. These additional conditions arose because of the fact that in the course of the proof, it was shown that u was either zero or harmonic at each point of D. It is clear then that we have the following theorem:

THEOREM 2. *If u is of class C' in D and if for each circle K in D with perimeter C we have*

$$\iint_K (u_x^2 + u_y^2)\,dxdy - \int_C u(\partial u/\partial n)\,ds = o(r^2),$$

then u is harmonic in D.

Let E be a bounded point set which has a conductor potential in the strict sense; that is, a function u continuous in the finite plane, harmonic except on E, equal to 1 on E, and with the appropriate logarithmic singularity at ∞. If u had continuous partial derivatives, Theorem 1 could be applied to this function $u - 1$ and u would be harmonic in the finite plane, which is impossible. Thus the following theorem may be stated.

THEOREM 3. *No conductor potential in the strict sense can have continuous partial derivatives.*

This is perhaps not surprising. The usual situation for smooth sets is for the conductor distributions of smooth sets to lie on curves, and for the discontinuity of the normal derivative across these curves to be proportional to the linear density. However for general point sets the result is not particularly obvious.

7. A theorem on analytic functions. We can prove a theorem analogous to Theorem 1 for analytic functions. In this case, however, the continuity of the derivatives is not required.

THEOREM 4. *Let $f(z)$ be continuous in a domain D, and at each point of D let f be either analytic or zero. Then f is analytic in D.*

The proof is very similar to that of Theorem 1 and details will be omitted. If $f = u + iv$, then in the analogue of Lemma 2, one proves that u_x, u_y are summable over D'. In the analogue of Lemma 4, one shows that u is absolutely continuous with respect to x and $u_x = 0$ almost everywhere in $D - D'$. To prove the special case of Theorem 4, it is shown that the integral of f around rectangles is zero, and Morera's theorem applies. In proving the general case of the theorem, Lemma 5 is applied directly to f.

Added in Proof. Professor G. R. MacLane has recently pointed out that Theorem 4 is known, and indeed is known as the Radó-Behnke-Stein-Cartan theorem. The contribution of Radó (*Über eine nicht fortsetzbare Riemannsche Mannigfaltigkeit,* Mathematische Zeitschrift, vol. 20 (1924), pp. 1-6) is a proof of our Lemma 5. The corresponding result for functions of several complex variables was proved by P. Thullen (*Uber die wesentlichen Singularitaten, usw.,* Mathematische Annalen, vol. 111 (1935), pp. 137-157). H. Behnke and K. Stein (*Modification komplexer Mannigfaltigkeiten und Riemanncher Gebiete,* Mathematische Annalen, vol. 124 (1951), pp. 1-16) proved our Theorem 4 for several complex variables. This amounts essentially to removing the hypothesis of simple connectedness from Lemma 5. Later and simpler proofs were given by H. Cartan (*Sur une extension d'un Théorème de Radó,* Mathematische Annalen, vol. 125 (1952-53), pp. 49-50) and E. Heinz (*Ein elementarer Beweis des Satzes von Radó-Behnke-Stein-Cartan,* Mathematische Annalen, vol. 131 (1956), pp. 258-259), using a variety of methods. The present method appears to be different from the previous one, and also yields the corresponding theorem for harmonic functions. Of course, the theorem for harmonic functions also follows from the theorem for analytic functions with the aid of our preliminary lemma.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES.

REFERENCES.

[1] E. F. Beckenbach, "On characteristic properties of harmonic functions," *Proceedings of the American Mathematical Society,* vol. 3 (1952), pp. 765-769.

[2] A. Ostrowski, "Quasianalytische Funktionen und asymptotische Entwickelung," *Acta Mathematica,* vol. 53 (1929), p. 251.

[3] L. Bieberbach, *Lehrbuch der Funktionentheorie,* Leipzig, 1934.

FUNDAMENTAL GROUPS ON A LORENTZ MANIFOLD.* †

By J. Wolfgang Smith.

---

1. Introduction. Let $\mathcal{V}$ denote a differentiable manifold of dimension $n \geqq 2$ and class C^p, $2 < p \leqq \infty$. A *metric Lorentz structure* $\mathcal{L}$ on $\mathcal{V}$ is a second order symmetric covariant tensor field of class C^q, $2 \leqq q \leqq p$, and signature $(2-n)$. Thus $\mathcal{L}$ is a quadratic form having 1 positive and $(n-1)$ negative eigenvalues everywhere on $\mathcal{V}$. It defines an inner product in the tangent vector spaces to $\mathcal{V}$ and enables us to distinguish *time-like*, *null* and *space-like* tangent vectors, according as their length squared is positive, zero or negative, respectively. The totality of null-vectors in a given tangent vector space constitutes a cone with vertex at the origin, sometimes referred to as a *light cone*. If one keeps the light cones but discards the metric tensor field, one obtains what may be called a *conformal Lorentz structure* L on $\mathcal{V}$.[1] We are interested in studying the *global character* of such a structure. The term *global character* has of course no precise meaning until one indicates a method whereby it may be measured or described. To give an example of one such method, we propose to regard L as a cross-section of an appropriate tensor bundle over $\mathcal{V}$. The homotopy class of the cross-section measures then a global characteristic of L.[2] In contradistinction to these purely *topological* properties, we shall be concerned with more refined global characteristics of L which are *not* invariant under arbitrary homotopies of the cross-section. Such characteristics may be called *conformal global properties* (c. g. p.) of the Lorentz structure. Perhaps the simplest example of a c. g. p. is furnished by the *barriers* defined in [7].

In the present paper we shall study certain c. g. p. which are described by means of an infinite collection of groups. The basic process employed to

---

* Received March 21, 1960.

† The results established in this paper have been announced under the same title in *Proceedings of the National Academy of Sciences, U. S. A.*, vol. 46 (1960), pp. 111-114. This research was supported in part by the United States Air Force under contract No. AF-49(638)-42, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

[1] $\mathcal{L}$ and L may be referred to as *associated* structures.

[2] This is essentially what the physicists Finkelstein and Misner [3] have done to obtain particle numbers for certain geons.

define these groups has been outlined in [8] and may be summarized as follows. Let x be a point of $\mathcal{V}$ and X a set of loops in $\mathcal{V}$ based at x. A homotopy $F: I \times I \to \mathcal{V}$ (where I denotes the closed unit interval) is said to be *in* X if the maps $F_t: I \to \mathcal{V}$, defined by the restriction of F to $I \times t$, belong to X for all $t \in I$. Two elements $f_0, f_1 \in X$ are *homotopic in* X (written $f_0 \sim f_1$ in X) if there exists a homotopy F in X with $F_i = f_i$ for $i = 0, 1$. Let ξ denote the set of equivalence classes in X under homotopy in X. In the event that a given multiplication in X induces a group structure on ξ we will say that ξ *arises from* X. Apart from the fundamental group $\pi_1(\mathcal{V}, x)$, numerous other groups arise from suitable loop spaces X. The groups τ_q in question have been obtained in [8] from certain loop spaces T_q, $q = 0, 1, 2, \cdots$; which depend on L.

We shall henceforth use the term *loop space* to denote a set of loops having a common base point, together with a law of multiplication, which gives rise to a group. Two loop spaces X_1, X_2 will be called *equivalent* if there exists a loop space X with $X_i \subset X$, such that the inclusion maps I_i: $X_i \to X$ induce isomorphisms of the corresponding groups. We agree always to identify groups arising from equivalent loop spaces. To obtain the groups τ_q, in particular, one has a choice of various equivalent loop spaces. In the present paper we shall take as our starting point certain loop spaces $T_q(L, x)$, to be defined in Section 2, which permit us to state the proof of basic results with optimum economy. The proof of equivalence for the $T_q(L, x)$ and the T_q considered in [8] is no more than an exercise, and will be omitted.

This work is part of a general effort to gain insight into the global structure of general Lorentz manifolds. Whatever interest the τ-groups may have in themselves, we believe that their significance must ultimately be judged in terms of their ability to contribute towards such an insight. It is too early to predict whether the new groups will be fruitful or not, in this sense. The most that can be claimed for them at this stage is that they bring to light some hitherto unrecognized global properties of a general Lorentz manifold. The present paper is devoted to the proof of four theorems concerning the τ-groups. The first (§ 4) establishes a general connection between the τ-groups and the fundamental group π_1 of $\mathcal{V}$. The second and third theorems (§ 5) are concerned with the dependence of the new groups on base point, and the last (§ 7) tells essentially what happens when the Lorentz structure is flat and complete.

2. The loop spaces. Our first task is to define the loop spaces $T_q(L, x)$. By a *path* we shall understand a piecewise differentiable map $f: I \to \mathcal{V}$, and

we will say that f *starts* at $f(0)$ and *ends* at $f(1)$. If f and g are two paths such that f ends where g starts, then one can define a product fg in the usual way,[3] giving a path which starts at $f(0)$ and ends at $g(1)$. The totality of paths in $\mathfrak{V}$, together with this law of multiplication, will be denoted by P^*. A path f, being piecewise differentiable, determines a unique tangent vector at $f(t)$ for every regular point $t \in I$, and two distinct tangent vectors for every singular $t \in I$. The path will be called *time-like* if all its tangent vectors are time-like. Two time-like vectors belonging to the same tangent vector space are called *oppositely* or *equally directed*, depending on whether they are separated or not by the light cone. A time-like path f is said to have an *interior corner* at $t \in I$ if t is a singular point of f, and if the corresponding tangent vectors to f at $f(t)$ are oppositely directed. It is said to have an *exterior corner* if it is closed, and if its initial and terminal tangent vectors are oppositely directed. We define a *q-loop based at x* (q a nonnegative integer) to be a time-like path which starts and ends at x and has exactly q corners.[4] Products involving q-loops based at x will be the basic ingredients of $T_q(L, x)$. These alone, however, would only give rise to a semi-group, and one must add some additional ingredients in order to obtain an identity and inverse elements. On the other hand, one must not add too much, for this would only lead to the usual fundamental group. We consequently define a *sting based at x* to be a path of the form ff^{-1} with $f(0) = x$. The set generated by q-loops and stings, based at x, does give rise to the desired group $\tau_q(L, x)$. It will be convenient, however, to use an equivalent loop space containing certain additional elements. For this purpose a few more definitions will be required. If f is an arbitrary path and $y \in f(I)$, we define a *y-factorization* of f to be one of the form f_+f_- with $f_+(1) = y$. If g is a path starting at y and $f^* = f_+gg^{-1}f_-$, we will say that f^* is obtained from f by *insertion of a sting*. Similarly, f is obtained from f^* by *deletion of a sting*. We define a *generalized q-loop* to be a path which can be obtained from a q-loop by insertion and deletion of stings (applied a finite number of times.) The constant path e_x (mapping I on x) will also be regarded as a generalized q-loop based at x, for all values of q. One observes that a p-loop f with $p \leqq q$ and p having the same *parity* as q (this means $p + q$ is even) is likewise a generalized q-loop. For clearly f can be turned into a q-loop by insertion of stings. The generalized q-loops are in a sense limits of q-loops, and it will be good to have them in our loop spaces. We consequently define $L_q^*(L, x)$ to be the set of all generalized q-loops based at x, and $T_q^*(L, x)$ to be the

[3] Cf., for instance, P. J. Hilton [4].

[4] This is a slightly weaker definition than the one given in [8].

subset of P^* generated by $L_q^*(L,x)$. In other words, the elements of $T_q^*(L,x)$ are finite products of generalized q-loops based at x. One verifies immediately that $T_q^*(L,x)$ is a loop space.

The asterisk indicates that we are still not satisfied. This stems from the fact that it will be greatly advantageous to deal exclusively with paths whose parametrization has been *standardized* in some way. For this purpose we will put a Riemannian metric[5] $\mathcal{R}$ on $\mathcal{V}$ and use arc length over total length (relative to $\mathcal{R}$) as standard parameter. One obtains thus a map S from P^* into the set P of all standardized paths. We define a multiplication in P to be the multiplication in P^* followed by S. The multiplication in P is thus associative. We further note the following properties of S which are easily verified:

(i) for all loops $f \in P^*$, $Sf \sim f$ in P^*;

(ii) for all $f \in T_q^*(L,x)$, $Sf \sim f$ in $T_q^*((L,x)$;

(iii) for $f, g \in P^*$ with fg defined, $S(fg) = S(f)S(g)$, the multiplication on the right being understood in P;

(iv) $f \sim g$ in X implies $Sf \sim Sg$ in $S(X)$.

Now let $T_q(L,x) = S(T_q^*(L,x))$. Clearly $T_q(L,x)$ is again a loop space (multiplication being understood in P), and it follows by properties (ii), (iii) and (iv) that $T_q(L,x)$ is equivalent to $T_q^*(L,x)$.[6] We take the sets $T_q(L,x)$, $q = 0, 1, 2, \cdots$, as the starting point of our investigation, and we will denote the corresponding groups by $\tau_q(L,x)$.[7]

It is known that the fundamental group $\pi_1(\mathcal{V},x)$ arises from the set $M^*(\mathcal{V},x)$ of all piecewise differentiable loops based at x, with the usual law of multiplication. It follows again from properties (i), (iii) and (iv) that the set $M(\mathcal{V},x)$ of all standardized loops based at x, with multiplication in P, is equivalent to $M^*(\mathcal{V},x)$. We can therefore obtain $\pi_1(\mathcal{V},x)$ from the loop space $M(\mathcal{V},x)$.

3. Notation and remarks. A little more terminology concerning paths will be helpful. If f, f^* are time-like paths with ff^* defined, we will say that

[5] For basic terminology and facts of differential geometry, in modern form, we refer to K. Nomizu [6].

[6] The inclusion map $I: T_q \to T_q^*$ induces a map i of the corresponding groups. Properties (ii) and (iii) imply that i is a homomorphism, (ii) implies that it is an epimorphism, and (iv) implies that it is a monomorphism.

[7] We will generally suppress the symbols (L,x) and write τ_q, etc. whenever the meaning is sufficiently clear.

f *joins smoothly* to f^*, provided the terminal tangent vector to f and the initial tangent vector to f^* are equally directed. Two paths f, f^* will be called *equal modulo stings* (written $f \doteq f^*$) if one can be obtained from the other by insertion and deletion of stings (applied a finite number of times). An element $f \in T_q(L, x)$ may be called *stingless* if there does not exist a path f^* (different from f) such that f is obtained from f^* by insertion of stings. It is not difficult to see that every $f \in T_q(L, x)$ is equal modulo stings to exactly one stingless element $f^* \in T_q(L, x)$. The operation producing f^* from f may be called *pruning*. Lastly, if f is a path and $t_1, t_2 \in I$, we will denote by $(f \mid t_1, t_2)$ the path g, defined by the formula

$$g(t) = f(t_1 + t(t_2 - t_1)), \qquad t \in I.$$

So far we have employed a Riemannian structure $\mathcal{R}$ only to standardize the parametrization of paths. Apart from this, we shall have occasion to utilize the fact that $\mathcal{R}$ determines a distance function Δ (mapping $\mathcal{V} \times \mathcal{V}$ into the set of nonnegative real numbers). For $x \in \mathcal{V}$ and ϵ a positive number, $\mathfrak{S}(x, \epsilon)$ will denote the totality of points $y \in \mathcal{V}$ with $\Delta(x, y) < \epsilon$. It will always be understood that ϵ has been chosen sufficiently small to insure that $\mathfrak{S}(x, \epsilon)$ is topologically an open ball (n-cell). In addition to the Riemannian structure $\mathcal{R}$, we shall sometimes utilize a metric Lorentz structure $\mathcal{L}$ associated with L. *Geodesics* and *parallel transport* will be understood relative to $\mathcal{L}$,[8] while *distance* and *path length* will be understood relative to $\mathcal{R}$. Our assumption concerning the differentiability class of $\mathcal{L}$ was designed to insure that, given $x \in \mathcal{V}$ and a nonzero tangent vector ξ at x, there shall exist a geodesic g which starts at x and has its initial tangent vector in the direction of ξ. Moreover, for ϵ sufficiently small, there exists exactly one such (standardized) g of length ϵ.

With every $f \in M(\mathcal{V}, x)$ we can associate a Lorentz transformation T_f of the tangent vector space $\mathcal{V}_x$ at x in the following manner. For $\xi \in \mathcal{V}_x$, $T_f\xi$ shall be the vector resulting from the parallel transport of ξ around f. If ξ is time-like, so is $T_f\xi$, and one may call T_f *time-preserving* or *time-reversing*, depending on whether ξ and $T_f\xi$ are equally or oppositely directed, respectively. It is not difficult to see that this property of T_f depends only on the homotopy class of f and on the conformal Lorentz structure L associated with $\mathcal{L}$. In the event that T_f is time-preserving for all $f \in M(\mathcal{V}, x)$, L may be called *time-orientable*. We also define the *parity* of $f \in M(\mathcal{V}, x)$ to be

[8] More precisely, we should say that there exists a unique torsionless affine connection which preserves $\mathcal{L}$. Geodesics and parallel transport will be understood in terms of this connection.

even or odd, depending on whether T_f is time-preserving or time-reversing. Let $f \in T_q(L,x)$, let $\xi \in \mathcal{V}_x$ be time-like, and assume that ξ and the initial tangent vector to f are equally directed. Let ξ_t denote the parallel translate of ξ along $(f \mid 0,t)$. If t is a nonsingular point of f, we note that ξ_t and the tangent vector to f at $f(t)$ are equally or oppositely directed, depending on whether the number of interior corners of $(f \mid 0,t)$ is even or odd. The parity of f is consequently equal to the parity of q.

The proofs given in the subsequent sections depend on certain elementary propositions of Lorentz geometry in the small which we now summarize in the form of three Lemmas. The first two refer to the local behavior of geodesics and can be stated as follows:

LEMMA 3.1. *Given $x \in \mathcal{V}$, there exists a positive number ϵ such that two points of $\mathfrak{S}(x,\epsilon)$ are joined by at most one geodesic lying wholly in $\mathfrak{S}(x,\epsilon)$.*

LEMMA 3.2. *If $x, y \in \mathcal{V}$ are joined by a unique time-like geodesic, then there exists a positive number ϵ such that every point of $\mathfrak{S}(x,\epsilon)$ is joined to y by a time-like geodesic.*

The third Lemma refers to the deformability of time-like paths. In our terminology it may be formulated as follows:

LEMMA 3.3. *Let f be a 0-loop based at x and let $f \in f(I)$. There exists then an open neighborhood $\mathcal{N}$ of y such that, for every $z \in \mathcal{N}$, there exists a 0-loop based at z. If g is an arbitrary path with $f(t) \in g(I)$ for an interior point $t \in I$, then there exists an open neighborhood $\mathcal{N}$ in $g(I)$ containing $f(t)$ such that f can be deformed as a 0-loop based at x to sweep over $\mathcal{N}$.*

The proofs of the three propositions offer no difficulty and will be omitted here.

4. Direct limits. The inclusion maps $I_q\colon T_q(L,x) \to M(\mathcal{V},x)$, $H_q\colon T_q(L,x) \to T_{q+2}(L,x)$ induce homomorphisms i_q, h_q of the corresponding groups, and we observe that the diagram

$$\begin{array}{ccc} \tau_q(L,x) & \xrightarrow{h_q} & \tau_{q+2}(L,x) \\ & {\scriptstyle i_q} \searrow \quad \swarrow {\scriptstyle i_{q+2}} & \\ & \pi_1(\mathcal{V},x) & \end{array} \tag{4.1}$$

is commutative. With every base point $x \in \mathcal{V}$ we have thus associated an algebraic structure which is summarized by diagram (4.1), drawn for

$q = 0, 1, 2, \cdots$. The first point to note is that the maps i_q and h_q are neither monomorphisms nor epimorphisms in general, as will be shown in Section 6 by means of examples. Thus even on a simply-connected manifold the τ-groups may assume nontrivial values. In order to find a connection between the τ-groups and the fundamental group π_1, one must consider the direct limits[9] of

$$\tau_0 \xrightarrow{h_0} \tau_2 \to \cdots \text{ and } \tau_1 \xrightarrow{h_1} \tau_3 \to \cdots.$$

Let us designate these direct limits by τ_+ and τ_-, respectively. Since diagram (4.1) is commutative, the maps i_q determine two homomorphisms $i_\pm : \tau_\pm \to \pi_1$.

THEOREM 4.1. *$i_+(i_-)$ maps $\tau_+(\tau_-)$ isomorphically onto the subgroup $\pi_1^+(\pi_1^-)$ of π_1, generated by loops of even (odd) parity. If L is time-orientable, $\pi_1^+ = \pi_1$ and π_1^-, τ_q (q odd) are trivial. Otherwise π_1 is a normal subgroup of index 2 and $\pi_1^- = \pi_1$.*

We begin the proof by defining sets

$$T_+ = \bigcup_{q \text{ even}} T_q, \qquad T_- = \bigcup_{q \text{ odd}} T_q.$$

We also let $M_+(M_-)$ denote the subset of M generated by loops of even (odd) parity. The purely geometrical content of the theorem may now be summarized in two propositions, as follows:

LEMMA 4.1. *Let $f_0, f_1 \in T_+(T_-)$. If $f_0 \sim f_1$ in M, then there exists an even (odd) integer q such that $f_0 \sim f_1$ in T_q.*

LEMMA 4.2. *Given $f_0 \in M_+(M_-)$, there exists $f_1 \in T_+(T_-)$ such that $f_0 \sim f_1$ in $M_+(M_-)$.*

These assertions are not at all surprising, but they are a little awkward to prove. Let us first consider Lemma 4.1. By assumption there exists a homotopy $F: I \times I \to \mathcal{V}$ in M with $F_i = f_i$ for $i = 0, 1$. We will show by a constructive argument that there exists an integer q and a homotopy $H: I \times I \to \mathcal{V}$ in T_q such that $H_i \sim F_i$ in T_q for $i = 0, 1$. The construction will depend on a Riemannian structure $\mathcal{R}$ and a metric Lorentz structure $\mathcal{L}$ associated with L. For $y \in F(I \times I)$ let $S(y)$ denote the set of all positive numbers ϵ such that (i) two points of $\mathfrak{S}(y, 2\epsilon)$ are connected by at most one geodesic lying wholly in $\mathfrak{S}(y, 2\epsilon)$,[10] (ii) $\epsilon \leqq 1$. By Lemma 3.1 $S(y)$ is not

[9] Cf., for instance, Eilenberg and Steenrod [2].

[10] In keeping with an agreement made in Section 3, we are taking ϵ sufficiently small to insure that $\mathfrak{S}(y, 2\epsilon)$ shall be an n-cell.

empty, and we let $\epsilon_1(y)$ denote its least upper bound. Since $\epsilon_1(y)$ is clearly lower semicontinuous, it assumes a smallest (positive) value ϵ_1 on $F(I \times I)$. Now let ξ be a time-like tangent vector at our base point x. With every point $(s,t) \in I \times I$ we associate a time-like tangent vector $\xi_{s,t}$ by parallel translating ξ along $(F_t \mid 0, s)$. There exists exactly one (standardized) geodesic $G_{s,t}$ which (i) starts at $F(s,t)$, (ii) has initial tangent vector in the direction of $\xi_{s,t}$, (iii) has total length ϵ_1. By Lemma 3.2 there exist positive numbers $\epsilon_{s,t}$, bounded above by ϵ_1, such that $F(s^*,t) \in \mathfrak{S}(F(s,t),\epsilon_{s,t})$ implies the existence of a unique geodesic $H^{s^*}{}_{s,t}$, lying wholly in $\mathfrak{S}(F(s,t),\epsilon_1)$, with

$$H^{s^*}{}_{s,t}(0) = G_{s,t}(1) \text{ and } H^{s^*}{}_{s,t}(1) = F(s^*,t).$$

We observe that the covering $\{\mathfrak{S}(F(s,t),\epsilon_{s,t}), (s,t) \in I \times I\}$ of $F(I \times I)$ admits a finite subcovering. There exists consequently a partition of I, given by $0 = s_0 < s_1 < \cdots < s_N = 1$, such that

$$F(s_{i+1},t) \in \mathfrak{S}(F(s_i,t),\epsilon_{s_i,t}), \qquad 0 \leqq i < N, t \in I.$$

We now define the desired homotopy H by the formula

$$H_t = \prod_{i=0}^{N-1} G_{s_i,t} H^{s_{i+1}}{}_{s_i,t}. \tag{4.1}$$

We further define homotopies G^0 and G^1 by

$$G_t{}^i = \prod_{j=0}^{N-1} G_{s_j,i} H^{u_j(t)}{}_{s_j,i}(F_i \mid u_j(t), s_{j+1}), \qquad i = 0, 1, \tag{4.2}$$

where

$$u_j(t) = s_j + t(s_{j+1} - s_j).$$

One observes that

$$G_0{}^i = F_i, \qquad G_1{}^i = H_i, \qquad i = 0, 1. \tag{4.3}$$

It remains now to prove that G^0, H and G^1 are homotopies in T_q for some q. For every $t \in I$, $G_t{}^0$, H_t and $G_t{}^1$ are time-like paths which start and end at x. We let $q^0(t)$, $p(t)$ and $q^1(t)$, respectively, denote the total number of corners. Since $F_0, F_1 \in T_+(T_-)$, there exists an even (odd) integer r such that $F_0, F_1 \in T_r$. In virtue of Equation (4.3) we can conclude that $q^0(t)$, $p(t)$ and $q^1(t)$ are even (odd) for all $t \in I$. Moreover, one finds that

$$p(t) \leqq 2N, \qquad q^i(t) \leqq p(t) + r \leqq 2N + r, \qquad i = 0, 1.$$

Consequently G^0, H and G^1 are homotopies in T_q with

$$q = \max_{t \in I} \{q^0(t), p(t), q^1(t)\},$$

as was to be proved. The proof of Lemma 4. 2 is accomplished by means of the same construction and need not be given here.

We proceed now to prove the theorem. Let $\tau_+^*(\tau_-^*)$ denote the group arising from $T_+(T_-)$. We shall show that $\tau_\pm^*$ is naturally isomorphic to $\tau_\pm$. The latter may be regarded as a set of equivalence classes in $S_\pm$, where

$$S_+ = \bigcup_{q \text{ even}} \tau_q, \qquad S_- = \bigcup_{q \text{ odd}} \tau_q,$$

two elements of $S_\pm$ being called *equivalent* if one can be mapped on the other by a composition of the maps h_q. For $a \in S_\pm$, let $\{a\}$ denote its equivalence class. We define now two maps $\phi_\pm : \tau_\pm^* \to \tau_\pm$ by the rule

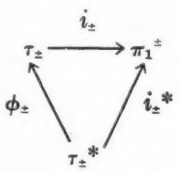
$$\phi_\pm([l]_\pm) = \{[l]_q\},$$

where l denotes an arbitrary element of $T_\pm$, $[l]_\pm$ denotes the corresponding element in $\tau_\pm^*$, q denotes the smallest integer p such that $l \in T_p$, and $[l]_q$ denotes the corresponding element of τ_q. To show that $\phi_\pm$ is well defined, the corresponding element of τ_q. To show that $\phi_\pm$ is well defined, we suppose $l_0 \sim l_1$ in $T_\pm$. Let q_i denote the smallest integer p such that $l_i \in T_p$, $i = 0, 1$. By Lemma 4.1 there exists an integer $q \geqq q_0, q_1$ such that $l_0 \sim l_1$ in T_q. Consequently

$$\phi_\pm([l_0]_\pm) = \{[l_0]_{q_0}\} = \{[l_0]_q\} = \{[l_1]_q\} = \{[l_1]_{q_1}\} = \phi_\pm([l_1]_\pm),$$

showing that $\phi_\pm$ is well defined. One verifies by similar elementary considerations that $\phi_\pm$ is an isomorphism, and that the diagram

(4.4)



(where $i_\pm^*$ denotes the homomorphism induced by the inclusion map $I_\pm^* : T_\pm \to M_\pm$) is commutative. We claim now that $i_\pm^*$ is an isomorphism. This is so because $i_\pm^*$ must be a monomorphism in virtue of Lemma 4.1, and it must be an epimorphism by Lemma 4.2. Since diagram (4.4) is commutative, we may conclude that $i_\pm$ is likewise an isomorphism.

If L is time-orientable, it is clear that $\pi_1^+ = \pi_1$, and that π_1^-, τ_q (q odd) are trivial. We suppose now that L is not time-orientable. For $l_+ \in M_+$ and $l \in M$, ll_+l^{-1} belongs again to M_+, implying that π_1^+ is normal. It remains to show that π_1^+ has index 2 in π_1, and that $\pi_1^- = \pi_1$. For this purpose we observe that the Riemannian structure $\mathfrak{R}$ permits us to distinguish a special

line element in every tangent vector space $\mathcal{V}_y$ to $\mathcal{V}$ as axis of the light cone. Each axis contains two unit vectors $\xi_y^{\pm}$, one being the negative of the other. The totality of pairs $(\xi_y^{\pm}, y)$, $y \in \mathcal{V}$, may be endowed with a manifold structure, and one thus obtains a manifold $\mathcal{V}^*$ covering $\mathcal{V}$. Since L is not time-orientable, $\mathcal{V}^*$ must be connected. We observe now[11] that $\pi_1(\mathcal{V}^*)$ is isomorphic to $\pi_1^+(\mathcal{V})$, and since $\mathcal{V}^*$ is a 2-fold covering of $\mathcal{V}$, $\pi_1^+(\mathcal{V})$ has index 2 in $\pi_1(\mathcal{V})$. Thus $\pi_1 = \pi_1^+ \cup \pi_1^*$, where π_1^* denotes a coset of π_1^+ in π_1. We note that $\pi_1^* \subset \pi_1^-$, since every element of π_1 which does not belong to π_1^- belongs to π_1^+. Let α be an arbitrary element of π_1 and $\beta \in \pi_1^*$. Then $\alpha\beta^{-1} \in \pi_1^*$, and consequently $\alpha\beta^{-1}$ and β both belong to π_1^-. But this implies $\alpha \in \pi_1^-$, as was to be proved.

5. Components of L. In contradistinction to $\pi_1(\mathcal{V}, x)$, the new groups $\tau_q(L, x)$ do in general depend on x as abstract groups.[12] We define a *component* of L to be a maximal connected subset $\mathcal{U}$ of $\mathcal{V}$ such that, for all $x, y \in \mathcal{U}$, there shall exist isomorphisms $\phi_q : \tau_q(L, y) \to \tau_q(L, x)$, $q \geqq 0$, which make the following diagram commutative:

$$\begin{array}{ccc} \tau_q(L,y) & \xrightarrow{\phi_q} & \tau_q(L,x) \\ \downarrow{\scriptstyle h_q} & & \downarrow{\scriptstyle h_q} \\ \tau_{q+2}(L,y) & \xrightarrow{\phi_{q+2}} & \tau_{q+2}(L,x). \end{array}$$

This definition implies that the totality of components constitutes a partition (disjoint covering) of $\mathcal{V}$. A Lorentz manifold with just one component may be called *simple*. If x is a point of $\mathcal{V}$, we will denote by $\{x\}$ the component to which it belongs. In this Section we shall establish a geometric characterization of components $\{x\}$ for which $\tau_0(L, x)$ is nontrivial, and we shall show in this case that $\pi_1^+(\{x\}, x)$ is a quotient group of $\tau_0(L, x)$.

THEOREM 5.1. *Let $\tau_0(L, x)$ be nontrivial for some $x \in \mathcal{V}$. Then $\{x\}$ is the totality of points in $\mathcal{V}$ belonging to the range of 0-loops based at x. Moreover, $\tau_0(L, z)$ is trivial for all $z \in \partial\{x\}$*[13].

Let $\mathcal{U}$ denote the totality of points in $\mathcal{V}$ belonging to the range of 0-loops based at x. It follows directly from Lemma 3.3 that every point of $\mathcal{U}$ different from x is an interior point of $\mathcal{U}$. To see that x itself is an interior point,

[11] Cf., also L. Markus [5], p. 412.

[12] Cf., for instance, Example A, Section 6.

[13] For $\mathcal{U} \subset \mathcal{V}$, $\partial\mathcal{U}$ shall denote the boundary of $\mathcal{U}$ in the sense of general topology.

we take a 0-loop f based at x and apply Lemma 3.3 to f^2. Hence $\mathcal{U}$ is open. If $\tau_0(L,z)$ is nontrivial for $z\in\partial\mathcal{U}$, then there exists a 0-loop l based at z. This implies by Lemma 3.3 that z has a neighborhood $\mathcal{N}$ such that, for every $y\in\mathcal{N}$, there exists a 0-loop l^* based at z with $y\in l^*(I)$. Since $z\in\partial\mathcal{U}$, we can take $y\in\mathcal{N}\cap\mathcal{U}$. If $l_+{}^*l_-{}^*$ is a y-factorization of l^*, $l_-{}^*l_+{}^*$ will be a 0-loop based at y. Since $y\in\mathcal{U}$, there exists a 0-loop f based at x with $y\in f(I)$. Let f_+f_- be a y-factorization of f. Then either $f_+l_-{}^*l_+{}^*f_-$ or else $f_-{}^{-1}l_-{}^*l_+{}^*f_+{}^{-1}$ must be a 0-loop based at x, which implies that $z\in\mathcal{U}$. On the other hand, since $\mathcal{U}$ is open, $z\notin\mathcal{U}$. Hence $\tau_0(L,z)$ is trivial. We conclude that $\{x\}\subset\mathcal{U}$.

It remains to show that $\mathcal{U}\subset\{x\}$. Let y be an arbitrary point of $\mathcal{U}$. There exists then a 0-loop l based at x with $y\in l(I)$. Let l_+l_- be a y-factorization of l, and let q be a nonnegative integer. We wish now to construct a map $\Phi\colon T_q(L,y)\to T_q(L,x)$ which will induce the desired isomorphism ϕ_q of the corresponding groups. To define $\Phi_q(f)$ for an arbitrary element $f\in T_q(L,y)$, we will first prune f to obtain a stingless element f^*. One must now distinguish three cases: (i) $f^*=e_y$, (ii) f^* is a p-loop (based at y) and joins smoothly to l_-, (iii) f^* is a p-loop and joins smoothly to $l_+{}^{-1}$. We define $\Phi_q(f)$ in each case as follows:

$$\Phi_q(f)=\begin{cases}\text{(i)} & l_+fl_+{}^{-1}\\ \text{(ii)} & l_+fl_-l^{-1}\\ \text{(iii)} & ll_-{}^{-1}fl_+{}^{-1}.\end{cases} \tag{5.1}$$

To prove that $\Phi_q(f)\in T_q(L,x)$, one verifies first that

$$\Phi_q(fg)\doteq\Phi_q(f)\Phi_q(g), \qquad f,g\in T_q(L,y); \tag{5.2}$$

in all cases. It follows by induction on the number ν that

$$\Phi_q\Big(\prod_{i=1}^{\nu}f_i\Big)\doteq\prod_{i=1}^{\nu}\Phi_q(f_i), \qquad f_i\in T_q(L,y). \tag{5.3}$$

Since $f\in T_q(L,y)$, it has a factorization

$$f=\prod_{i=1}^{\nu}f_i, \qquad f_i\in L_q(L,y).^{14} \tag{5.4}$$

In virtue of Equations (5.3) and (5.4), it remains to prove that $\Phi_q(f_i)\in T_q(L,x)$ for $f_i\in L_q(L,y)$. This is certainly true if f_i places us in case (i) as defined above. In case (ii), $l_+f_il_-$ is clearly an element of $L_q(L,x)$, and it remains to verify that $l^{-1}\in T_q(L,x)$. To show this, it suffices to observe that $l^{-1}\doteq(l_+f_il_-)\,(ll_+f_il_-)^{-1}$, and that $ll_+f_il_-\in L_q(L,x)$. Similar observations show that $\Phi_q(f_i)\in T_q(L,x)$ in case (iii), and the map $\Phi_q\colon T_q(L,y)\to T_q(L,x)$ is consequently well defined.

[14] $L_q=S(L_q{}^*)$, as defined in Section 2.

It is not difficult to see that $f \sim g$ in $T_q(L, y)$ implies $\Phi_q(f) \sim \Phi_q(g)$ in $T_q(L, x)$. Therefore Φ_q induces a map $\phi_q\colon \tau_q(L, y) \to \tau_q(L, x)$. Equation (5.2) implies that ϕ_q preserves multiplication, and since $\Phi_q(e_y) = l_+ l_+^{-1}$, ϕ_q also preserves the identity. Hence ϕ_q is a homomorphism. To show that it is an isomorphism, we define a map $\Psi_q\colon T_q(L, x) \to T_q(L, y)$ by the formula

$$\Psi_q(f) = \begin{cases} \text{(i)} & l_+^{-1} f l_+ \\ \text{(ii)} & l_+^{-1} l_-^{-1} l_- f l_+ \\ \text{(iii)} & l_+^{-1} f l_-^{-1} l_- l_+, \end{cases} \tag{5.5}$$

where cases (i), (ii) and (iii) are defined just as above, with $l_+(l_-^{-1})$ playing the role of $l_-(l_+^{-1})$. One verifies as before that Ψ_q is well defined, and that it induces a homomorphism $\psi_q\colon \tau_q(L, x) \to \tau_q(L, y)$. A simple calculation based on Equations (5.1) and (5.5) discloses that $\psi_q \circ \phi_q$ and $\phi_q \circ \psi_q$ are identity maps, which proves that ψ_q is an isomorphism.[15]

It remains now to check that the maps ϕ_q commute with the maps h_q. But this follows from the fact that the diagram[16]

$$\begin{array}{ccc} T_q(L, y) & \xrightarrow{\Phi_q} & T_q(L, x) \\ \downarrow & & \downarrow \\ T_{q+2}(L, y) & \xrightarrow{\Phi_{q+2}} & T_{q+2}(L, x) \end{array}$$

is clearly commutative.

Having concluded the proof of Theorem 5.1, we proceed to prove

THEOREM 5.2. *If $\tau_0(L, x)$ is nontrivial, then there exists a natural epimorphism $k\colon \tau_0(L, x) \to \pi_1^+(\{x\}, x)$.*

If f is an arbitrary (nontrivial) element of $T_0(L, x)$, we can prune it to obtain an element f^* which is a product of 0-loops based at x. By the preceding theorem, $f^*(I) \subset \{x\}$. We have consequently defined a map $K\colon T_0(L, x) \to M^+(\{x\}, x)$, and this induces a homomorphism $k\colon \tau_0(L, x) \to \pi_1^+(\{x\}, x)$. Our main task is to verify that an arbitrary element $f \in M^+(\{x\}, x)$ is homotopic in $M^+(\{x\}, x)$ to a product of 0-loops based at x.

In virtue of Theorem 5.1 there exists, for every $s \in I$, a 0-loop f_s based

---

[15] To indicate how this calculation goes, we take $f \in T_q(L, y)$ and place ourselves in case (ii). Then

$$\Psi_q \circ \Phi_q(f) = l_+^{-1} l_-^{-1} l_- (l_+ f l_- l_-^{-1} l_+^{-1}) l_+ \doteq f.$$

The five remaining computations are quite similar.

[16] The vertical arrows signify inclusion maps.

at x and an interior point $t_s \in I$ such that $f_s(t_s) = f(s)$. It follows by Lemma 3.3 that every $s \in I$ has an open neighborhood N_s in I, an associated continuous family $\{F_u^s, u \in N_s\}$ of 0-loops based at x, and an associated continuous function $u_s\colon N_s \to I$ such that $F_t^s(u_s(t)) = f(t)$ for $t \in N_s$, $F_s^s = f_s$ and $u_s(s) = t_s$. One sees by elementary considerations that there exists a finite partition of I, given by $0 = s_0 < s_1 < \cdots < s_\nu = 1$, such that $s_{i-1} \in N_{s_i}$ for $0 < i \leqq \nu$.

We can now simplify our notation by setting $f_i = f_{s_i}$, $F_i = F^{s_i}_{s_{i-1}}$, $t_i = t_{s_i}$ and $u_i = u_{s_i}(s_{i-1})$, $0 < i \leqq \nu$. Thus $F_{i+1}(u_{i+1}) = f_i(t_i)$, $0 < i < \nu$, and we may suppose that our loops have been so oriented that $(F_{i+1} \mid 0, u_{i+1})$ does *not* join smoothly to $(f_i \mid t_i, 1)$. We may suppose further that $(F_1 \mid u_1, 1)$ is a 0-loop based at x. Let

$$g_\mu = \prod_{i=1}^{\mu} (F_i \mid u_i, 1)(f_i \mid 1, t_i), \qquad 0 < \mu \leqq \nu. \tag{5.6}$$

It is not difficult to see, in the first place, that $g_\nu \sim f$ in $M^+(\{x\}, x)$. The formula required to prove this is entirely analogous to Equation (4.2) and need scarcely be written down. One observes further that g_ν is a product involving 0-loops based at x, and possibly some 1-loops based at x with an exterior corner. If the latter do not occur, the result is proved. In the general case we will show that g_ν can be modified by the insertion of stings to yield a product of 0-loops based at x.

For this purpose we posit the inductive hypothesis

$$g_i \doteq l_i m_i (f_i \mid 1, t_i), \qquad 0 < i < \mu \leqq \nu, \tag{5.7}$$

where l_i denotes a product of 0-loops based at x and m_i is a closed time-like path (without interior corners) which joins smoothly to $(f_i \mid 1, t_i)$, or else $m_i = e_x$. Our assumption concerning $(F_1 \mid u_1, 1)$ insures the validity of this inductive hypothesis. We claim now that a corresponding formula holds for $i = \mu$. This is certainly the case when

$$l = m_{\mu-1}(f_{\mu-1} \mid 1, t_{\mu-1})(F_\mu \mid u_\mu, 1)$$

is a 0-loop, for one can then take $l_\mu = l_{\mu-1} l$ and $m_\mu = e_x$. Otherwise l is 1-loop with exterior corner, and $l_{\mu-1} l \in M^-(\{x\}, x)$. This being the case, we will establish the existence of a closed time-like path m such that lm is a 0-loop based at x. Let us suppose for the moment that this has been done. One can then set $l_\mu = l_{\mu-1} lm$ and $m_\mu = m^{-1}$, and it is easily checked that m_μ joins smoothly to $(f_\mu \mid 1, t_\mu)$, as is required. This completes the inductive argument, and one can conclude that Equation (5.7) holds for $i = \nu$. The

factor $m_\nu(f_\nu \mid 1, t_\nu)$ must be a 0-loop or a 1-loop. Since $l_\nu, g_\nu \in M^+(\{x\}, x)$, the second possibility is ruled out, as was to be proved.

It remains now to construct the path m referred to above. If $\mu = \nu$, then the loop

$$g_\mu{}^* = l_{\mu-1} l(f_\mu \mid 0, t_\mu)$$

must belong to $M^+(\{x\}, x)$, since it differs from g_ν only by a 0-loop and a sting. Consequently $l(f_\mu \mid 0, t_\mu)$ must be a 0-loop, and we can take $m = (f_\mu \mid 0, t_\mu)$. We suppose now $\mu < \nu$. If $\mu + 1 = \nu$, then one observes again that the loop

$$g_{\mu+1}{}^* = l_{\mu-1} l(f_\mu \mid 0, t_\mu)\,(F_{\mu+1} \mid u_{\mu+1}, 0)\,(f_{\mu+1} \mid 1, t_{\mu+1})$$

must belong to $M^+(\{x\}, x)$, since it differs from g_ν only by 0-loops and stings. In this case we may therefore take

$$m = (f_\mu \mid 0, t_\mu)\,(F_{\mu+1} \mid u_{\mu+1}, 0)\,(f_{\mu+1} \mid 1, t_{\mu+1}).$$

If, on the other hand $\mu + 1 < \nu$, then we may continue our construction in the obvious way. After at most $(\nu - \mu)$ steps we will have obtained a suitable path m.

As a corollary to Theorem 5.2, we note that if $\mathcal{V}$ is simple, then τ_0 is either trivial or it admits $\pi_1{}^+$ as a quotient. In particular, if $\mathcal{V}$ is simple and compact, then τ_0 admits $\pi_1{}^+$ as a quotient.[17]

6. Examples.

(A) The simplest example of a Lorentz manifold is obtained by endowing the real number plane R^2 with a metric tensor, whose components relative to the standard coordinate system are given by

$$\| g_{ij} \| = \begin{Vmatrix} +1 & 0 \\ 0 & -1 \end{Vmatrix}.$$

Clearly all the τ-groups for this space are trivial. Now let us remove a closed space-like line segment yy^* from R^2 and consider the resulting manifold $\mathcal{V}$. It is clear that all groups τ_q (q odd) are trivial, because the given Lorentz structure L is time-orientable. It is also evident that τ_0 is everywhere trivial, and one sees without much difficulty that τ_q is everywhere infinite cyclic for $q = 4, 6, 8 \cdots$. This is due to the fact that, given $x \in \mathcal{V}$, there exists a 4-loop based at x which encircles the missing interval yy^*. The situation is different in regards to τ_2. A given null-line through y will intersect a null-line through y^* at some point z. A second null-line through y will intersect a null-line

[17] We do not presently know whether there exist compact Lorentz manifolds which are not simple.

through y^* at some point z^*. We claim now that $\tau_2(L, x)$ is trivial for all points x belonging to the closed triangular region $y\,z\,y^*$ (or $y\,z^*\,y^*$). For there does not exist a 2-loop based at x which encircles yy^*. For all other points $x \in \mathcal{V}$, such a 2-loop does exist, and one finds again that $\tau_2(L, x)$ is infinite cyclic. This example shows that the maps h_q and i_q are not epimorphisms in general.

(B) For our second example we take $\mathcal{V}$ to be the 3-dimensional number space R^3. A conformal Lorentz structure L may be defined by prescribing a smooth vector field η on R^3. Since R^3 carries a natural Euclidean structure, η determines in every tangent vector space $\mathcal{V}_x$ a right circular cone with vertex at the origin and axis parallel to $\eta(x)$. We take this to be our light cone. Let r, θ, z be cylindrical coordinates in R^3, and let $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ denote the corresponding local reference frame. We shall set

$$\eta = a_\theta \vec{e}_\theta + a_z \vec{e}_z,$$

where

$$a_z = 1 - a_\theta$$

$$a_\theta = f(r),$$

f being a monotonic C^∞ function such that

$$f(t) = \begin{cases} 0 & t \leqq 0 \\ 1 & t \geqq 1. \end{cases}$$

The resulting Lorentz structure evidently admits 0-loops. In particular, if $x = (1, 0, 0)$, then the equations

$$r = 1$$

$$\theta = 2\pi t \qquad 0 \leqq t \leqq 1$$

$$z = 0$$

represent a 0-loop based at x. This implies that $\tau_0(L, x)$ is nontrivial. It is extremely likely that the given Lorentz manifold has exactly one component with τ_0 infinite cyclic, all other groups being trivial.

(C) For our last example we take $\mathcal{V}$ to be the portion of R^3 defined by $|z| < \epsilon$, where ϵ is a positive number. Let x_1, x_2, x_3 denote the natural coordinate numbers in R^3 (with $x_3 = z$) and let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ denote the corresponding local reference frame. We set

$$\eta = a_2 \vec{e}_2 + a_3 \vec{e}_3$$

with

$$a_3 = 1 - a_2$$

$$a_2 = f(r-1).$$

Let $x = (3,0,0)$, $x' = (0,4,0)$, $x'' = (-3,0,0)$ and $x''' = (0,-4,0)$. The The ploygonal line $xx'x''x'''x$ represents now a 2-loop based at x, with corners at x' and x'''. We assert that if ϵ is sufficiently small, then this 2-loop represents a nontrivial element of $\tau_2(L,x)$. For to collapse it, one must make it intersect the line $r=0$ (the x_3-axis). But it is easily verified that a timelike path which starts and ends at a point with $r>1$ and intersects the line $r=0$ cannot have fewer than $(1+1/\epsilon)$ corners. This example shows that the maps i_q and h_q are not monomorphisms in general.

A remark concerning *completeness*[18] is in order at this point. The τ-groups and the associated homomorphisms depend only on the conformal Lorentz structure L. Completeness, on the other hand, is a property of the metric Lorentz structure $\mathfrak{L}$, and is not a conformal invariant. We believe that completeness by itself is quite irrelevant to the conformal global properties in question. In its stead one requires a purely conformal concept of normality which insures, at least, that the Lorentz manifold is not a fragment of a larger one. For Lorentz structures on the plane, one can define *normality* to mean the absence of certain conformal global objects (called *barriers* in [7]), and one finds (i) two arbitrary normal Lorentz structures are conformally isometric, (ii) every normal Lorentz structure is geodesically connected, (iii) if $\mathfrak{L}$ is complete and has an absolutely convergent curvature integral, then the associated conformal structure L is normal. This concept of normality can easily be generalized to arbitrary Lorentz manifolds, and it may turn out to be useful. In particular we do expect that normality, in this sense, will impose some restrictions on the behavior of the τ-groups.

7. Lorentz manifolds which are flat and complete. In this Section it will be assumed that L has an associated metric Lorentz structure $\mathfrak{L}$ which is flat and complete. More precisely, this means that the torsionless affine connection canonically associated with $\mathfrak{L}$ is flat and complete.[19] As one may expect, these conditions have a strong implication concerning the τ-groups. The basic result can be stated as follows:

[18] Cf. Section 7.

[19] *Flat* means that the curvature vanishes, and *complete* means that all geodesics are extendible to infinite values of a canonical parameter. Cf. Nomizu [6].

THEOREM 7.1. *If $\mathcal{V}$ is flat and complete,*[20] *then the maps h_0 are monomorphisms and the maps h_q, $q > 0$, are isomorphisms.*

Let $\mathcal{V}^*$ denote the universal covering space of $\mathcal{V}$, endowed with the metric Lorentz structure induced by $\mathcal{V}$. Let $\mathcal{L}^n$ denote what may be called *standard Lorentz space* of dimension n. It is obtained by endowing the number space R^n with a metric tensor, whose components relative to the standard coordinate system are given by

$$\| g_{ij} \| = \begin{Vmatrix} 1 & 0 \cdots & 0 \\ 0 & -1 \cdots & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 \cdots & -1 \end{Vmatrix}.$$

Since $\mathcal{V}^*$ is flat, complete and simply connected, there exists[21] a global isometry $\phi: \mathcal{V}^* \to \mathcal{L}^n$. We will henceforth identify $\mathcal{V}^*$ with $\mathcal{L}^n$.

To prove that $h_q: \tau_q(\mathcal{V}, x) \to \tau_{q+2}(\mathcal{V}, x)$ is a monomorphism for $q \geqq 0$ and $x \in \mathcal{V}$, we must show that $f \not\sim e_x$ in $T_q(\mathcal{V}, x)$ implies $f \not\sim e_x$ in $T_{q+2}(\mathcal{V}, x)$, for $f \in T_q(\mathcal{V}, x)$. It suffices to show that $f \sim e_x$ in $M(\mathcal{V}, x)$ implies $f \sim e_x$ in $T_q(\mathcal{V}, x)$. Let us therefore suppose that f is null-homotopic. If $y_0 \in \mathcal{V}^*$ lies above x, then f lifts to an element $f^* \in T_q(\mathcal{V}^*, y_0)$. Now let $H_t: \mathcal{V}^* \to \mathcal{V}^*$ denote the homothety map defined by the formula[22]

$$H_t(y) = y_0 + t(y - y_0) \tag{7.1}$$

for $t \in I$ and $y \in \mathcal{V}^*$. Clearly $H_t \circ f^*$ defines a homotopy in $T_q(\mathcal{V}^*, y_0)$, connecting f^* and e_{y_0}. Let π denote the projection from $\mathcal{V}^*$ to $\mathcal{V}$. Then $\pi \circ H_t \circ f^*$ defines a homotopy in $T_q(\mathcal{V}, x)$, connecting f and e_x.

It remains to show that the maps h_q are epimorphisms for $q > 0$. Given $f \in T_{q+2}(\mathcal{V}, x)$, we must show that there exists $g \in T_q(\mathcal{V}, x)$ such that $f \sim g$ in $T_{q+2}(\mathcal{V}, x)$. It suffices to consider the case where f is a $(q+2)$-loop. If $f \sim e_x$ in $T_{q+2}(\mathcal{V}, x)$, the result is trivial. We suppose therefore that $f \not\sim e_x$ in $T_{q+2}(\mathcal{V}, x)$, which implies (by the result just proved) that f is not null-homotopic. The lift f^* of f, determined by the base point y_0, will consequently end at a point y^* different from y_0. Clearly there exists a polygonal line of time-like segments $l^* = y_0 y_1 \cdots y_r y^*$ in $\mathcal{V}^*$, with noncollinear vertices, such

[20] In this Section it will be convenient to let $\mathcal{V}$ denote the differentiable manifold, together with $\mathcal{L}$.

[21] Cf. Ehresmann [1].

[22] To interpret Equation (7.1), we recall that $\mathcal{V}^*$ is identified with $\mathcal{L}^n$, which has a natural vector space structure.

that $f \sim \pi \circ l^*$ in $T_{q+2}(\mathcal{V}, x)$. One may further assume that the points y_0, $y_{\nu-1}$, y_ν and y^* are coplanar. We take ν to be minimal and assert that $\nu = 0$ or 1. For let L denote the line through y_ν and y^*, and suppose $\nu > 0$. The segment $H_t \circ (y_{\nu-1} y_\nu)$, $0 < t \leqq 1$, can be extended to meet L at a point z_t. As t varies from 1 to 0, z_t varies from y_ν to a limiting position z_0. Clearly y^* cannot lie between y_ν and z_0 on L, for this would contradict the minimality of ν. Consequently

$$\pi \circ \{[H_t \circ (y_0 y_1 \cdot \cdot \cdot y_\nu)][H_t(y_\nu) z_t y^*]\}$$

defines a homotopy in $T_{q+2}(\mathcal{V}, x)$ connecting $\pi \circ l^*$ with $\pi \circ (y_0 z_0 y^*)$. Thus if $\nu > 0$, one may conclude that $\nu = 1$. $\pi \circ l^*$ is therefore a p-loop, the possible values of p being 0, 1, and 2. If q is odd, $p = 1$ and $\pi \circ l^* \in T_q(\mathcal{V}, x)$. Otherwise $p = 0$ or 2, and again $\pi \circ l^* \in T_q(\mathcal{V}, x)$. This proves that the maps h_q are epimorphisms for $q > 0$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

REFERENCES.

[1] Ch. Ehresmann, "Les connexions infinitesimales dans un espace fibré différentiable," *Colloque de Topologie* (Espaces Fibrés), Bruxelles, 1950.

[2] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952.

[3] D. Finkelstein and C. W. Misner, "Some new conservation laws," *Annals of Physics*, vol. 6 (1959), pp. 230-243.

[4] P. J. Hilton, *An introduction to homotopy theory*, Cambridge University Press, 1953.

[5] L. Markus, "Line element fields and Lorentz structures on differentiable manifolds," *Annals of Mathematics*, vol. 62 (1955), pp. 411-417.

[6] K. Nomizu, *Lie groups and differential geometry*, Mathematical Society of Japan, 1956.

[7] J. W. Smith, "Lorentz structures on the plane," *Transactions of the American Mathematical Society*, vol. 95 (1960), pp. 226-237.

[8] ———, "Fundamental groups on a Lorentz manifold," *Proceedings of the National Academy of Sciences, U. S. A.*, vol. 46 (1960), pp. 111-114.

ANALYTIC AND ALGEBRAIC DEPENDENCE OF MEROMORPHIC FUNCTIONS.*

By REINHOLD REMMERT.†

---

1. Let V be an irreducible, locally irreducible complex space[1] and $m_1, \cdots, m_k, m$ meromorphic functions on V such that m is analytically dependent on $m_1, \cdots, m_k$. It is a classical problem to find sufficient conditions under which such a dependence is an algebraic one. A well-known theorem states that analytic dependence always entails algebraic dependence if V is compact (compare [5], [13]). In his papers [10], [12] W. Thimm pointed out another sufficient condition expressed in terms of the structure of the set of indeterminacy of the functions $m_1, \cdots, m_k$. Here we prove a theorem which covers the results of [5], [13] as well as those of [10], [12].

In order to make things clearer we first recall some basic notions. If m_1 is a meromorphic function on V, then the set A_1 of zeros of m_1 and the set B_1 of poles of m_1 is well defined, and both are analytic sets in V which are empty or of codimension 1 everywhere.[2] The intersection $N_1 := A_1 \cap B_1$ is called the set of indeterminacy of m_1, N_1 is an analytic subset of V of codimension at least 2. If $\dim V > 1$, in general N_1 is not empty. For a finite system $m_1, \cdots, m_k$ of meromorphic functions on V we denote by N_κ the set of indeterminacy of m_κ, $\kappa = 1, \cdots, k$, and call $N := \bigcup_{\kappa=1}^{k} N_\kappa$ the set of indeterminacy of the system $m_1, \cdots, m_k$; obviously N is an analytic set in V of codimension at least 2. We denote by S_k the complex product manifold of k complex projective lines and consider the holomorphic map $m_1 \times \cdots \times m_k : V - N \to S_k$ defined by $v \to (m_1(v), \cdots, m_k(v))$, $v \in V - N$. This map cannot be continued to a holomorphic map of the total space V

---

* Received April 5, 1960.

† Part of this paper was developed during the summer 1959 while the author was at the University of Michigan, Ann Arbor.

[1] For the definition of complex spaces compare [2].

[2] By dimension and codimension of complex spaces (resp. analytic sets) we always mean complex dimension and codimension. We denote by $\dim_x X$ the (complex) dimension of a complex space X at a point $x \in X$ and write $\dim X = \max_{x \in X} \dim_x X$. The topological dimension of a topological space X will be denoted by $\dim_R X$.

into S_k, nevertheless for any point $v \in N$ the image set $m_1 \times \cdots \times m_k(v)$ is well defined: it consists of all points $s \in S_k$ which are cluster points of sequences $m_1 \times \cdots \times m_k(v_n)$, where $v_n \in V - N$ tends to v. In this way the meromorphic functions $m_1, \cdots, m_k$ give rise to the correspondence

$$v \to m_1 \times \cdots \times m_k(v) \subset S_k, v \in V,$$

which is called the meromorphic map defined by $m_1, \cdots, m_k$. For any set $M \subset V$ we define the image $m_1 \times \cdots \times m_k(M)$ to be the set

$$\bigcup_{v \in M} m_1 \times \cdots \times m_k(v).$$

For convenience we repeat the definitions of analytic and algebraic dependence. A meromorphic function m on V is called *analytically dependent* on the meromorphic functions $m_1, \cdots, m_k$ if there exist a nonempty domain $U \subset V$ and a holomorphic function $f(w, z_1, \cdots, z_k)$ in a domain G of the space C^{k+1} of the variables $w, z_1, \cdots, z_k$ such that the following is true: $f'_w(w, z_1, \cdots, z_k) \not\equiv 0$; the restrictions $m \mid U, m_1 \mid U, \cdots, m_k \mid U$ are holomorphic in U; $m \times m_1 \times \cdots \times m_k(U) \subset G$; $f(m(v), m_1(v), \cdots, m_k(v)) = 0$ for each point $v \in U$.

m is called *algebraically dependent* on $m_1, \cdots, m_k$ if there exists a polynomial $p(w, z_1, \cdots, z_k)$ with $p'_w(w, z_1, \cdots, z_k) \not\equiv 0$ such that $p(m(v), m_1(v), \cdots, m_k(v)) = 0$ for each point $v \in V$ at which all the functions $m, m_1, \cdots, m_k$ are holomorphic.

If K is any set in V, the notion of a germ of a meromorphic function in K is well defined. Let us denote the set of all germs of meromorphic functions in K by R_K. Obviously R_K forms a ring with respect to the natural addition and multiplication of germs. In the following we always assume K to be connected, in which case R_K is a field. The notions of analytic and algebraic dependence carry over in a natural way to germs of meromorphic functions.

If $m_1, \cdots, m_k$ are meromorphic functions in an open set U of V, we denote their corresponding germs on a given connected set $K \subset U$ by $\tilde{m}_1, \cdots, \tilde{m}_k$. These germs generate a subfield of R_K which is usually denoted by $C(\tilde{m}_1, \cdots, \tilde{m}_k)$. The set L_K of all germs of meromorphic functions on K which depend analytically on $\tilde{m}_1, \cdots, \tilde{m}_k$ obviously forms a subfield of R_K containing $C(\tilde{m}_1, \cdots, \tilde{m}_k)$; we call L_K the *analytic closure* of $C(\tilde{m}_1, \cdots, \tilde{m}_k)$ in R_K. We are interested in the algebraic structure of this field L_K. In particular, we want to derive a condition insuring that L_K is an algebraic extension of the field $C(\tilde{m}_1, \cdots, \tilde{m}_k)$. We prove the following

THEOREM. *Let K be a connected compact analytic set in an irreducible,*

locally irreducible complex space V with a countable base of open sets, and let $m_1, \cdots, m_k$ be meromorphic functions in a neighborhood U of K such that $\dim_R(m_1 \times \cdots \times m_k(U)) = \dim_R(m_1 \times \cdots \times m_k(K))$.[2] *Let us denote by $\tilde{m}_1, \cdots, \tilde{m}_k$ the germs of meromorphic functions in K defined by $m_1, \cdots, m_k$ on K. Then the analytic closure L_K of $C(\tilde{m}_1, \cdots, \tilde{m}_k)$ in R_K coincides with the algebraic closure of $C(\tilde{m}_1, \cdots, \tilde{m}_k)$ in R_K and is a simple algebraic extension of $C(\tilde{m}_1, \cdots, \tilde{m}_k)$.*[3]

Let us explicitly state the following

COROLLARY. *Let $m_1, \cdots, m_k$ be meromorphic functions on an irreducible, locally irreducible*[3] *complex space V; let $K \subset V$ be a compact analytic set such that* $\dim_R(m_1 \times \cdots \times m_k(K)) = \dim_R(m_1 \times \cdots \times m_k(V))$. *Then the field L of all meromorphic functions on V which depend analytically on $m_1, \cdots, m_k$ is a simple algebraic extension of the field $C(m_1, \cdots, m_k)$.*

In particular this corollary tells us that under the condition $m_1 \times \cdots \times m_k(K) = m_1 \times \cdots \times m_k(V)$ analytic dependence on $m_1, \cdots, m_k$ always entails algebraic dependence on $m_1, \cdots, m_k$. If V is compact then the condition of the corollary is fulfilled for $K = V$ and we have the results of [5], [13]. For the case when K is a single point $v_0 \in V$ the above theorem is essentially due to W. Thimm (compare [12], p. 156, verallgemeinerter Normalfall), who gave a rather complex proof. The following very special case of the corollary was also proved by K. Stein [8]:[4]

If m_1 is a meromorphic function on an irreducible, locally irreducible[3] *complex space V having at least one point of indeterminacy (i.e. N_1 is not empty), then any meromorphic function on V, which depends analytically on m_1, depends algebraically on m_1.*

In the following sections we give a simple proof of the above theorem (and its corollary) by using some results concerning holomorphic maps. The crucial points of the proof are a general lemma on holomorphically dependent holomorphic maps, which is proved in Section 2, and a well-known theorem of W. L. Chow [1] stating that any analytic set in S_k is algebraic.[5]

2. By X, Y, Z we always denote finite dimensional complex spaces with

[3] The condition that V is locally irreducible is only for convenience. If V is not locally irreducible one can always pass to the normalisation of V.

[4] Recently I was informed by K. Stein that he was able to prove the general theorem of Thimm by applying general results of his theory of complex base spaces (unpublished).

[5] For a simple proof of Chow's theorem see [6].

countable bases of open sets. Let $\eta: X \to Y$ be a holomorphic map. The rank $r_x(\eta)$ of η at a point $x \in X$ is defined to be the integer $\dim_x X - \dim_x \eta^{-1}(\eta(x))$, i. e. the codimension at x of the fibre $\eta^{-1}(\eta(x))$. The number $r(\eta) := \max_{x \in X} r_x(\eta)$ is called the (total) *rank* of η on X. Obviously we have $r(\eta) \leqq \dim X$. In [4], [7] it was proved that:

(a) *The image $\eta(X)$ is a topological space of topological dimension $2r(\eta)$. If X is compact, $\eta(X)$ is an analytic set in Y.*

From now on X is supposed to be irreducible. Besides $\eta: X \to Y$ we consider a second holomorphic map $\zeta: X \to Z$. We denote the map $x \to (\eta(x), \zeta(x))$ by $\eta \times \zeta$; obviously $\eta \times \zeta: X \to Y \times Z$ is also a holomorphic map. As $(\eta \times \zeta)^{-1}((\eta \times \zeta)(x)) = \eta^{-1}(\eta(x)) \cap \zeta^{-1}(\zeta(x))$ for all $x \in X$, we have $r_x(\eta) \leqq r_x(\eta \times \zeta)$ for all $x \in X$ and therefore especially $r(\eta) \leqq r(\eta \times \zeta)$. *The map ζ is called holomorphically dependent*[6] *on η if* $r(\eta) = r(\eta \times \zeta)$. This notion of holomorphic dependence of holomorphic maps generalises the well-known notion of analytic dependence of holomorphic functions. For it can easily be proved that:

(b) *A holomorphic function f on X depends analytically on a system $f_1, \cdots, f_k$ of holomorphic functions on X if and only if the equation* $r(f_1 \times \cdots \times f_k) = r(f_1 \times \cdots \times f_k \times f)$ holds for the holomorphic maps $f_1 \times \cdots \times f_k: X \to C^k$, $f: X \to C^1$.

Now we prove the following

LEMMA. *Let $\eta: X \to Y$, $\zeta: X \to Z$ be holomorphic maps of an irreducible complex space X into complex spaces Y, Z; let W be a subset of X such that* $\dim_R(\eta(W)) = \dim_R(\eta(X))$. *Suppose ζ depends holomorphically on η and $\eta \times \zeta(W)$ is an analytic set in $Y \times Z$. Then $\eta \times \zeta(X) = \eta \times \zeta(W)$.*

Proof. We have to prove $\tau^{-1}(\tau(W)) = X$, if we put $\tau := \eta \times \zeta$. Evidently $X' := \tau^{-1}(\tau(W))$ is an analytic set in X containing W. X being irreducible, it is sufficient to prove $\dim X' = \dim X$. Suppose $\dim X' < \dim X$. We consider the restriction maps $\eta' := \eta \mid X'$, $\zeta' := \zeta \mid X'$, $\tau' := \tau \mid X'$. As $\tau' = \eta' \times \zeta'$ and all fibres of τ' are also fibres of τ we get:

$$r_x(\eta') \leqq r_x(\tau') = \dim_x X' - \dim_x \tau'^{-1}(\tau'(x)) < \dim_x X - \dim_x \tau^{-1}(\tau(x)) = r_x(\tau)$$

for all $x \in X'$ and hence $r(\eta') < r(\tau)$. Now $r(\tau) = r(\eta)$ because ζ depends holomorphically on η. Therefore it follows $r(\eta') < r(\eta)$ and by using (a):

[6] This definition is due to K. Stein [9].

$\dim_R(\eta'(X')) < 2r(\eta)$. But this is impossible in view of $\eta'(X') = \eta(X') \supset \eta(W)$ and $\dim_R(\eta(W)) = 2r(\eta)$, q. e. d.

3. For the proof of our theorem we shall use the fact that the points of indeterminacy of a system of meromorphic functions can be removed by a modification. More exactly we shall need the following proposition:

(c) *For each system $m_1, \cdots, m_k$ of meromorphic functions on an irreducible, locally irreducible complex space V having the set N as set of indeterminacy there exists an irreducible, locally irreducible complex space V' and a proper surjective holomorphic map $\nu: V' \to V$ with the following properties:*

α) The topological closure $\bar{V}$ of the graph of the holomorphic map $m_1 \times \cdots \times m_k \,|\, V - N \to S_k$ in $V \times S_k$ is an irreducible analytic set in $V \times S_k$. V' is the normalisation space of $\bar{V}$, ν is the natural projection of V' onto V.

β) ν maps $V' - \nu^{-1}(N)$ biholomorphically onto $V - N$ and induces an isomorphism $\nu^*: k(V) \to k(V')$ of the field $k(V)$ of meromorphic functions on V onto the field $k(V')$ of meromorphic functions on V'. ($\nu^*(m) := m \circ \nu$ for each $m \in k(V)$.)

γ) The functions $\nu^*(m_1), \cdots, \nu^*(m_k)$ have no points of indeterminacy in V'.

δ) For each system $g_1, \cdots, g_r \in k(V)$ and each set $M \subset V$ we have

$$g_1 \times \cdots \times g_r(M) = \nu^*(g_1) \times \cdots \times \nu^*(g_r)\,(\nu^{-1}(M)).$$

The statements α)-γ) are proved in [4]; the proof of δ) runs as follows: We write for abbreviation $g := g_1 \times \cdots \times g_r$, $g' := \nu^*(g_1) \times \cdots \times \nu^*(g_r)$ and show first that $g(M) \subset g'(\nu^{-1}(M))$. Suppose $s \in g(v)$, $v \in M$. If we denote by N_g the set of indeterminacy of the functions $g_1, \cdots, g_r$, then by definition there exists a sequence $v_n \in V - N_g$ tending to v such that s is a cluster point of the sequence $g(v_n)$. For each n we choose a point $v'_n \in \nu^{-1}(v_n)$; as ν is a proper map, the sequence $v'_n \in V' - \nu^{-1}(N_g)$ has at least one cluster point v' in V'. Obviously $v' \in \nu^{-1}(v)$. Now we have $g' = g \circ \nu$ on $V' - \nu^{-1}(N_g)$, and therefore $g'(v'_n) = g(v_n)$. This shows that s is a cluster point of $g'(v'_n)$; i.e. $s \in g'(v')$ or $s \in g'(\nu^{-1}(M))$.

The proof of $g'(\nu^{-1}(M)) \subset g(M)$ follows in the same way.

A simple consequence of (a) and (c) is

(c′) *For each compact analytic set K in V the set $m_1 \times \cdots \times m_k(K) \subset S_k$ is algebraic.*

Proof. The map $m' : V' \to S_k$ is holomorphic ($m' := \nu^*(m_1) \times \cdots \times \nu^*(m_k)$). Therefore, $\nu^{-1}(K)$ being compact, $m'(\nu^{-1}(K))$ is analytic in S_k by virtue of (a). From (c) and Chow's theorem we then conclude that $m'(\nu^{-1}(K)) = m_1 \times \cdots \times m_k(K)$ is algebraic in S_k.[7]

4. In this section we prove first our theorem and then its corollary. Obviously we may assume $V = U$ without loss of generality. First we show that the field L_K is the algebraic closure of $C(\tilde{m}_1, \cdots, \tilde{m}_k)$ in R_K. To prove this we have only to show that each $\tilde{m} \in L_K$ is algebraic over $C(\tilde{m}_1, \cdots, \tilde{m}_k)$. We fix an element $\tilde{m} \in L_K$ and choose a meromorphic function m in a suitable connected neighbrohood T of K which induces $\tilde{m}$ on K and depends analytically on $m_1 \mid T, \cdots, m_k \mid T$. Then we are through if we show that m depends algebraically on $m_1 \mid T, \cdots, m_k \mid T$. From (c′) it follows easily that this statement is contained in the following

PROPOSITION. *Let $m_1, \cdots, m_k$ be a system of meromorphic functions on an irreducible, locally irreducible*[8] *complex space V and let K be a compact analytic set in V such that*

$$\dim_R(m_1 \times \cdots \times m_k(K)) = \dim_R(m_1 \times \cdots \times m_k(V)).$$

Then for any meromorphic function m in a connected neighborhood T of K depending analytically on $m_1 \mid T, \cdots, m_k \mid T$ we have:

$$m_1 \times \cdots \times m_k \times m(T) = m_1 \times \cdots \times m_k \times m(K);$$

$$\dim_R(m_1 \times \cdots \times m_k \times m(T)) = \dim_R(m_1 \times \cdots \times m_k(T)).$$

Proof. We apply (c) to the system $m_1 \mid T, \cdots, m_k \mid T, m$ defined on the irreducible and locally irreducible complex space T. We denote the modification by T' and the modification map by $\nu : T' \to T$ and write for abbreviation $\mu' := \nu^*(m_1 \mid T) \times \cdots \times \nu^*(m_k \mid T)$, $m' := \nu^*(m)$. The maps $\mu' : T' \to S_k$ and $m' : T' \to S_1$ are both holomorphic, and by assumption m' depends holomorphically on μ'. The set $K' := \nu^{-1}(K)$ is compact and analytic in T'; therefore by virtue of (c′), $\mu' \times m'(K')$ is an analytic set in S_{k+1}. As moreover we have $\dim_R(\mu'(K')) = \dim_R(\mu'(T'))$, we are able to apply the lemma of Section 2 (put $X := T'$, $Y := S_k$, $Z := S_1$, $W := K'$, $\eta := \mu'$, $\zeta := m'$). We conclude that $\mu' \times m'(K') = \mu' \times m'(T')$, or what is the same:

[7] This was already proved in [11], if K is a single point.

$$m_1 \times \cdots \times m_k \times m(T) = m_1 \times \cdots \times m_k \times m(K).$$

As

$$\dim_R(m_1 \times \cdots \times m_k \times m(T)) = 2r(\mu' \times m'),$$

$$\dim_R(m_1 \times \cdots \times m_k(T)) = 2r(\mu') \text{ and } r(\mu' \times m') = r(\mu'),$$

the second equation of the proposition follows immediately. Q. E. D.

It remains to prove that the field L_K is a simple extension of $C(\tilde{m}_1, \cdots, \tilde{m}_k)$. A theorem of classical algebra tells us that to prove this it is enough to show that there exists an integer $b > 0$ such that

$$(*) \qquad [\tilde{m} : C(\tilde{m}_1, \cdots, \tilde{m}_k)] \leqq b \text{ for all } \tilde{m} \in L_K.$$

We may assume without loss of generality that the set of indeterminacy of the system $m_1, \cdots, m_k$ is empty.[8]. For otherwise we just apply (c) and prove our statement for the fields

$$L_{K'} := \nu^*(L_K), \; C(\tilde{m}'_1, \cdots, \tilde{m}'_k) := \nu^*(C(\tilde{m}_1, \cdots, \tilde{m}_k)),$$

which are isomorphic to L_K and $C(\tilde{m}_1, \cdots, \tilde{m}_k)$ respectively. We denote the holomorphic map $m_1 \times \cdots \times m_k$ of V into S_k by μ and consider the set $B := \mu(V) \subset S_k$. The lemma of Section 2—applied to the map $\eta := \mu$ and a constant map ζ (with $W = K$)—immediately gives $\mu(V) = \mu(K)$; therefore the set B is an irreducible complex subspace of S_k. Obviously no set $\mu^{-1}(s) \cap K$, $s \in B$, is empty. As K is compact, there are only a finite number of components of each fibre $\mu^{-1}(s)$, $s \in B$, which intersect K. Let us denote this finite number by $n(s)$. A well-known proposition of Osgood ([3], p. 230) states the existence of an integer $b > 0$ and of a nonempty domain $G \subset B$ such that the set $G_1 := \{s \in G, n(s) \leqq b\}$ is dense in G. We claim that b is a number fulfilling the conditions of inequality (*).

For the proof let $\tilde{m} \in L_K$ be arbitrary. We choose a connected neighborhood T of K and a representative meromorphic function m of $\tilde{m}$ in T such that the conditions of our proposition are fulfilled. Then the set $A := (\mu \mid T \times m)(T) \subset S_k \times S_1$ is an irreducible complex subspace of $S_k \times S_1$. The restriction of the natural projection $S_k \times S_1 \to S_k$ to A defines a holomorphic surjective map $\pi : A \to B$. From the equation $\dim A = \dim B$ it follows that there exists a set N analytic and nowhere dense in B such that all sets $\pi^{-1}(s)$, $s \in B - N$, consist of a finite number of points.[9] This shows

[8] The following proof runs along almost the same lines as the proof of Satz 7 in the author's paper [5].

[9] Obviously N can be chosen as the π-image of the degeneration set of the map $\pi : A \to B$.

that the space $A - \pi^{-1}(N)$ is an "analytic covering space" of $B - N$ with respect to the map π.[10] Therefore the inequality $[\tilde{m}: C(\tilde{m}_1, \cdots, \tilde{m}_k)] \leqq b$ is equivalent to the fact that the number of sheets of this covering does not exceed b.

In order to prove this we denote the set of indeterminacy of m in T by M. All sets $\mu^{-1}(s) \cap M$ are empty for $s \in B - N$, as otherwise for a certain $s \in B - N$ the set $\pi^{-1}(s) = (\mu \mid T \times m)(\mu^{-1}(s) \cap T)$, which coincides with $(\mu \mid T)(\mu^{-1}(s) \cap T) \times m(\mu^{-1}(s) \cap T)$, would not be finite.[11] From the algebraic dependence of m on $m_1 \mid T, \cdots, m_k \mid T$ we now conclude that m is constant on each connected component of $\mu^{-1}(s) \cap T$, $s \in B - N$. Therefore it follows from $\mu \times m(K) = \mu \times m(T)$ that m has at most $n(s)$ different values, including ∞, on $\mu^{-1}(s) \cap T$, if $s \in B - N$. Therefore m has always at most b different values on the fibres $\mu^{-1}(s) \cap T$, $s \in G' - N$. But as $G' - N$ is everywhere dense in G, it follows immediately that for each s of the domain $G \subset B$ the set $\pi^{-1}(s)$ consists of at most b different points. This means exactly that the complex space $A - \pi^{-1}(N)$, considered as a covering space of $B - N$ with respect to π, has at most b sheets. Therefore the inequality (*) and hence the theorem is proved.

The proof of the corollary is now very simple. We may assume K to be irreducible. For the analytic set $m_1 \times \cdots \times m_k(V)$, being the holomorphic image of an irreducible complex space, is irreducible, and so there exists an irreducible component K_1 of K such that

$$m_1 \times \cdots \times m_k(K_1) = m_1 \times \cdots \times m_k(V).$$

We then have a natural (injective) field homomorphism of L into L_K which maps $C(m_1, \cdots, m_k)$ onto $C(\tilde{m}_1, \cdots, \tilde{m}_k)$. The field L therefore may be considered as a subfield of L_K containing $C(\tilde{m}_1, \cdots, \tilde{m}_k)$. As L_K is a simple algebraic extension of $C(\tilde{m}_1, \cdots, \tilde{m}_k)$ by our theorem, L is itself a simple algebraic extension of the field $C(m_1, \cdots, m_k)$, Q. E. D.

THE UNIVERSITY OF MICHIGAN
AND
INSTITUTE FOR ADVANCED STUDY.

---

[10] More precisely, the normalisation of $A - \pi^{-1}(N)$ is an analytic covering space of the normalisation of $B - N$. For the notion of analytic covering space compare [2].

[11] The identity $(\mu \mid T \times m)(D) = (\mu \mid T)(D) \times m(D)$ holds for each subset D of T, as $\mu \mid T$ is a holomorphic map.

REFERENCES.

[1] W. L. Chow, "On compact analytic varieties," *American Journal of Mathematics*, vol. 71 (1949), pp. 893-914.

[2] H. Grauert and R. Remmert, "Komplexe Räume," *Mathematische Annalen*, vol. 136 (1958), pp. 245-318.

[3] W. F. Osgood, *Lehrbuch der Funktionentheorie II, 1*, Berlin u. Leipzig, 1929.

[4] R. Remmert, "Holomorphe und meromorphe Abbildungen komplexer Räume," *Mathematische Annalen*, vol. 133 (1957), pp. 328-370.

[5] ———, "Meromorphe Funktionen in kompakten komplexen Räumen," *ibid.*, vol. 132 (1956), pp. 277-288.

[6] ——— and K. Stein, "Über die wesentlichen Singularitäten analytischer Mengen," *ibid.*, vol. 126 (1953), pp. 263-306.

[7] ———, "Eigentliche holomorphe Abbildungen," *Mathematische Zeitschrift*, vol. 73 (1960), pp. 159-189.

[8] K. Stein, "Analytische Projektion komplexer Mannigfaltigkeiten," *Coll. Fonct. Plus. Var.*, Brussels, 1953.

[9] ———, "Analytische Zerlegungen komplexer Räume," *Mathematische Annalen*, vol. 132 (1956), pp. 63-93.

[10] W. Thimm, "Über ausgeartete meromorphe Abbildungen II," *ibid.*, vol. 125 (1952), pp. 264-283.

[11] ———, "Über die Menge der singulären Bildpunkte einer meromorphen Abbildung," *Mathematische Zeitschrift*, vol. 57 (1953), pp. 456-480.

[12] ———, "Untersuchungen über ausgeartete meromorphe Abbildungen," *Mathematische Annalen*, vol. 127 (1954), pp. 150-161.

[13] ———, "Meromorphe Abbildungen von Riemannschen Bereichen," *Mathematische Zeitschrift*, vol. 60 (1954), pp. 435-457.

HEREDITARILY COMPACT SPACES.*

By A. H. Stone.

---

1. Introduction. By definition, a *hereditarily compact* space, or a "Zariski" or "Noether" space, is a topological space all of whose subspaces are compact.[1] Such spaces have received some attention [9, 10] because they arise in algebraic geometry (in the Zariski topology) and in some other algebraic constructions. Here we study these spaces on their own account. In the applications they are usually T_1 but not T_2; in fact, a T_2 hereditarily compact space is necessarily finite. However, we do not assume any separation axioms except where they are explicitly stated. We begin by giving some alternative characterizations (§ 2), and considering some properties related to some of them (§ 3). In § 4 we associate to every hereditarily compact space a topologically invariant ordinal number, its "type"; this corresponds to the dimension in the application to algebraic geometry. This permits the "construction" of all hereditarily compact spaces (§ 5). In § 6 we discuss the effect of various standard operations on such spaces on their types, and in § 7 we consider the countable hereditarily compact spaces in more detail.

Notation. A space X is *discrete* if each point $p \in X$ has a neighborhood in X consisting of p itself; X is *weakly discrete* if each $p \in X$ has a neighhood in X consisting of a finite set of points. (For T_1 spaces these notions are equivalent.) An indexed family $\{U_\lambda\}$ of subsets of X is called *finite* if $U_\lambda = \emptyset$ for all but finitely many values of λ.

2. Characterizations. We begin by observing that, in the definition of hereditary compactness, it is not necessary to specify that *all* subspaces are compact, and moreover the kind of compactness considered does not greatly matter. Incidentally we also obtain some further characterizations.

THEOREM 1. *The following statements about an arbitrary space X are equivalent:*

---

* Received April 25, 1960.

[1] Throughout this paper, "compact" means "quasicompact" in the sense of Bourbaki; that is, every open covering has a finite subcovering.

(1) *Every subspace of X is compact.*

(1_c) *Every countable subspace of X is compact.*

(1_o) *Every open subspace of X is compact.*

(2), (2_c), (2_o) *Every subspace (or countable, or open subspace) of X is sequentially compact.*

(3), (3_c), (3_o) *Every subspace (or countable, or open subspace) of X is countably compact.*

(4) *X has no weakly discrete infinite subspace.*

(5) *Every strictly decreasing sequence of closed subsets of X is finite.*

(6) *X has a sub-base $\mathfrak{B}$ of open sets such that every strictly increasing sequence of finite unions of members of $\mathfrak{B}$ is finite.*

Remark. The equivalence of (1), (1_o) and (5) is known (see [10] and Exposé 1 (by P. Cartier) of the Séminaire C. Chevalley, vol. 1, 1956-8).

Proof. Because countable compactness is implied by compactness or sequential compactness, it is enough to prove the implications (3_c) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1), (6) $\Rightarrow$ (5), (3_o) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2). All are easy; by way of example we prove (4) $\Rightarrow$ (5). If $C_1 \supset C_2 \supset \cdots$ is an infinite strictly decreasing sequence of closed subsets of X, pick $p_n \in C_n - C_{n+1}$ $(n = 1, 2, \cdots)$; the points p_n are all distinct, so the set $P = \{p_n\}$ is infinite. But each p_n has the neighborhood $P \cap (X - C_{n+1})$ in P, and this consists of the n points $p_1, \cdots, p_n$ only. Thus (4) is contradicted.

Consider now the following modified compactness conditions (all weaker than compactness) on a space X. (The list could easily be extended.)

(A) Every open covering of X has a finite subsystem whose union is dense in X.

(B_1) Every locally finite system of open sets in X is finite.

(B_2) Every locally finite system of disjoint open sets is finite.

(B_3) Every countable covering of X has a finite subsystem whose union is dense in X.

(C_1) Every locally finite open covering of X has a finite subcovering.

(C_2) Every countable locally finite open covering of X has a finite subcovering.

(D_1) Every locally finite open covering of X has a finite dense subsystem.

(D_2) Every countably infinite open covering of X has a proper dense subsystem.

(E_1) Every star-finite open covering of X is finite.

(E_2) Every countably infinite open covering of X by sets each of which meets at most two others has a proper dense subsystem.

(F) Every continuous real-valued function on X is bounded.

Property (B_2) is "feeble compactness" [7]; (B_1) has been called "light compactness" [1]; (F) is "pseudocompactness" [2]. It is easily seen that each of these properties implies the next, and that (B_1), (B_2), (B_3) are equivalent (see [1, 6]), and similarly (C_1) and (C_2), (D_1) and (D_2), and (E_1) and (E_2) are equivalent; it can be shown by examples that there are no other implications between them in general.[2] All except (A) are implied by countable compactness, and are equivalent to it for normal T_1 spaces [2], but not in general. Weaker separation axioms suffice for some other equivalences (for instance, regularity makes (B)—(E) equivalent [3, 4, 5]). But the hereditary forms of all these properties are equivalent, irrespective of separation axioms, as the next theorem shows.

THEOREM 2. *The following statements about an arbitrary space X are equivalent:*

(1) (1_c) *Every subspace (or countable subspace) of X has property* (A).

(2) (2_c) *Every subspace (or countable subspace) of X has property* (F).

(3) *X has no infinite discrete subspace.*

Further, if X is T_1, these statements are equivalent to the statements in Theorem 1.

It follows, of course, that any of the properties (B_1)—(E_2) could replace (A) or (F) here.

Proof. As (A) $\Rightarrow$ (F), it is enough to prove (2_c) $\Rightarrow$ (3) $\Rightarrow$ (1) and that if X is T_1 then (3) implies property (4) of Theorem 1. The first and last of these are trivial; to prove the second, suppose Y is a subspace of X which

[2] Even for T_1 spaces. An example having property (D) but not (C) is given in [1, p. 502] (note, however, that the statement on p. 503 lines 6, 7 is incorrect). It can be modified to give an example satisfying (E) but not (D). The example given at the beginning of § 3 below has property (A) without being countably compact. The usual space of countable ordinals is countably compact, and so satisfies (B), but does not satisfy (A). A suitable union of a sequence of spaces, each of which has no non-constant continuous function, will satisfy (F) but not (E). A T_0 space satisfying (C) but not (B) (from which a T_1 example can be derived by standard technique) is the set of all finite non-empty sets F of positive integers, in which each F has the single neighborhood $U(F)$ = family of non-empty subsets of F. For the properties discussed here (and many others), see [3, 4, 5, 6].

does not have property (A), and let $\{U_\lambda\}$ be an open covering of Y. If no finite subsystem of $\{U_\lambda\}$ is dense in Y, pick $z_1 \in Y$, say $z_1 \in U_{\lambda_1}$; pick $z_2 \in Y - \bar{U}_{\lambda_1}$, say $z_2 \in U_{\lambda_2}$, and generally pick $z_n \in Y - (\bar{U}_{\lambda_1} \cup \cdots \cup \bar{U}_{\lambda_{n-1}})$, say $z_n \in U_{\lambda_n}$. Put $V_n = U_{\lambda_n} \cap Y - (\bar{U}_{\lambda_1} \cup \cdots \cup \bar{U}_{\lambda_{n-1}})$, an open set containing z_n. The sets $V_1, V_2, \cdots$, are disjoint, so $Z = \{z_n\}$ is an infinite discrete subset of Y, contradicting (3).

Remark. In Theorem 2, in contrast to Theorem 1, it is not enough to require that every *open* subspace of X has the relevant properties, even if X is T_1. This is shown by the following example. Let X be the union of two disjoint infinite sets Y, Z; a subset of X is to be open if it is $\emptyset$ or contains all but finitely many points of Z. Then X is a T_1 space and every open subspace of X has property (A), but the closed subspace Y of X is discrete and does not even have property (F).

Further, the T_0 axiom (instead of T_1) would not suffice for the equivalence of the statements in Theorems 1 and 2. For let X be the space of positive integers, with $\emptyset$, X and the sets $\{1, 2, \cdots, n\}$ $(n = 1, 2, \cdots)$ as the only open sets. Every subset of X has property (A), but X is not compact.

3. Irreducibility. A space X is *irreducible* if it is not the union of two proper closed subsets; equivalently, every two non-empty open subsets of X intersect. It is known [10] that a hereditarily compact space is always expressible as the union of a finite number of irreducible sets. Here we amplify this property. We say that a space X is *semi-irreducible* if every family of disjoint (non-empty) open subsets of X is finite. Thus every hereditarily compact space is semi-irreducible; but the converse is false, even for T_1 spaces. (Take, for example, X to be an uncountable set in which the closed sets are X and its countable subsets; X is T_1 and irreducible but not even countably compact.) We note the following easily verified properties:

(1) If $A \subset X$, A is irreducible, or semi-irreducible, if and only if $\bar{A}$ has the corresponding property.

(2) If X is irreducible, or semi-irreducible, then so is every open subset of X.

(3) If X is semi-irreducible and non-empty, then X contains a non-empty maximal open irreducible subspace, and also a non-empty maximal irreducible subspace (which must be closed, from (1)).

(4) X is hereditarily irreducible if and only if the open sets of X are linearly ordered by inclusion; if X is T_1, it is hereditarily irreducible if and only if it has at most one point.

THEOREM 3. *The following statements about an arbitrary space X are equivalent:*

(i) *X is semi-irreducible.*

(ii) *There is a finite system of disjoint open irreducible subspaces $U_1, \cdots, U_n$ of X such that $\bigcup \bar{U}_i = X$.*

(iii) *X is the union of a finite number of disjoint irreducible subspaces, each the difference between two closed sets.*

(iv) *X is the union of a finite number of closed irreducible subspaces.*

(v) *X is the union of a finite number of semi-irreducible subspaces.*

(vi) *There is an integer N such that X does not contain more than N disjoint non-empty open sets.*

(vii) *X has only finitely many regular open sets.*[3]

Proof. (i) $\Rightarrow$ (ii) By Zorn's lemma there is a maximal system $\mathfrak{U}$ of disjoint open irreducible subsets of X; from (i), this system is finite, say $\mathfrak{U} = \{U_1, \cdots, U_n\}$. Let $V = X - \bigcup \bar{U}_i$; from (2) and (3) above, if $V \neq \emptyset$, V contains a non-empty open irreducible subset U_{n+1}, contradicting the maximality of $\mathfrak{U}$. Hence $V = \emptyset$ and $X = \bigcup \bar{U}_i$.

(ii) $\Rightarrow$ (iii) Put $Y_i = \bar{U}_i - \bigcup \{\bar{U}_j \mid j < i\}$ $(1 \leqq i \leqq n)$; then $U_i \subset Y_i \subset \bar{U}_i$, so $\bar{Y}_i = \bar{U}_i$ and Y_i is irreducible by (1) above. Since $X = \bigcup Y_i$, (iii) follows.

(iii) $\Rightarrow$ (iv) If $X = \bigcup Y_i$, where Y_i is irreducible, then $X = \bigcup \bar{Y}_i$, where $\bar{Y}_i$ is irreducible.

(iv) $\Rightarrow$ (v) trivially, because every irreducible space is semi-irreducible.

(v) $\Rightarrow$ (vi) Say $X = X_1 \cup \cdots \cup X_n$, where each X_i is semi-irreducible. Because (i) implies (iii), each X_i is the union of a finite number of irreducible sets, so we may write $X = Y_1 \cup \cdots \cup Y_N$, where each Y_i is irreducible. Suppose that $U_1, \cdots, U_{N+1}$ are disjoint non-empty open subsets of X. Each U_i meets some $Y_{j(i)}$, and we must have $j(i_1) = j(i_2)$ for two distinct integers i_1, i_2 (between 1 and $N+1$). Thus we may assume that both U_1 and U_2 meet Y_1; but this contradicts the irreducibility of Y

The implication (vi) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (vii) Let V be any regular open set in X; we show V is one of the 2^n interiors of unions of the sets $\bar{U}_i$ in (ii). We may suppose V meets $U_1, \cdots, U_r$ and is disjoint from $U_{r+1}, \cdots, U_n$ (where $0 \leqq r \leqq n$). Then, if $i \leqq r$, the closure $\mathrm{Cl}(V \cap U_i)$ of $V \cap U_i$ in X must be $\bar{U}_i$; for, as U_i is irreducible, the non-empty open set $V \cap U_i$ is dense in U_i. Hence

[3] A set G is "regular open" if and only if $G = \mathrm{Int}\,(\bar{G})$.

$$V = \mathrm{Int}(\bar{V}) = \mathrm{Int}(\bigcup \mathrm{Cl}(V \cap U_i)) = \mathrm{Int}(\bar{U}_1 \cup \cdots \cup \bar{U}_r).$$

(vii) $\Rightarrow$ (i) If X has an infinite family of disjoint (non-empty) open sets $G_1, G_2, \cdots$, the sets $X - \bar{G}_1, X - \bar{G}_2, \cdots$ provide infinitely many distinct regular open sets.

COROLLARY 1. *If X is semi-irreducible (a fortiori if X is hereditarily compact), X has only a finite number of components.*

For an irreducible space is connected.

COROLLARY 2. *If X is regular, X is semi-irreducible if and only if X has only finitely many open sets.*[4]

For, in a regular space, every open set is a union of regular open sets.

Remarks. (a) From (iv) of Theorem 3, we can write any semi-irreducible space X as $X_1 \cup \cdots \cup X_n$, where each X_i is closed and irreducible, and where no X_i is contained in any other. It is easy to see that the sets $X_1, \cdots, X_n$ are then uniquely determined, apart from their order. (Cf. [10] for the hereditarily compact case.)

(b) A connected semi-irreducible T_1 space need not be irreducible.

THEOREM 4. *For any Hausdorff space X, the following assertions are equivalent:*

(I) *X is hereditarily compact.*

(II) *X is semi-irreducible.*

(III) *X is finite.*

For trivially (I) implies (II) and (III) implies (I); that (II) implies (III) follows from Theorem 3(iv) since an irreducible Hausdorff space can have at most one point.

THEOREM 5. *The following statements about an arbitrary space X are equivalent to those in Theorem 2, and thus to the hereditary compactness of X if X is T_1:*

(1) *Every subspace of X is semi-irreducible.*

(1_c) *Every countable subspace of X is semi-irreducible.*

For if every subspace of X has property (A) (§ 2), it is clearly semi-

[4] "Regular" means that each point has a basis of closed neighborhoods; the T_1 axiom is not assumed. In the hereditarily compact case, Corollary 2 is due to Nollet [9].

irreducible. Conversely, if every countable subspace of X is semi-irreducible, X can contain no infinite discrete subspace.

Remark. The analogous statement (1_0)—that every *open* subspace of X is semi-irreducible—would not be equivalent to the statements in Theorem 5 in general, being equivalent to the semi-irreducibility of X.

4. The type of a hereditarily compact space. Let X be a hereditarily compact space, fixed for the moment. We assign, to each closed subspace of X, an ordinal number, its "type," as follows. The empty set (exceptionally) has type -1. When all the closed subsets of X of types $< \alpha$ have been dealt with, and if X has other closed subsets, then (by Theorem 1(5)) X has *minimal* closed subsets not of type $< \alpha$; each of these is said to be *irreducibly of type* α. The finite unions of sets irreducibly of type $\leqq \alpha$ are said to be of type $\leqq \alpha$, and a (closed) set of type $\leqq \alpha$ which is not of type $< \alpha$ is of type α. Ultimately all closed subsets of X (including X) are assigned types. The sets which are irreducibly of type 0 are precisely the non-empty *trivial* closed subsets of X; if X is T_1 they are the 1-point sets.[5] (Further examples will be given later.) The following two properties follow at once from the definitions.

(1) If $Y_1, \cdots, Y_n$ are closed subsets of X of types $\leqq \alpha$, then $Y_1 \cup \cdots \cup Y_n$ is of type $\leqq \alpha$.

(2) If Y is closed in X and is irreducibly of type α, then every closed proper subset Z of Y is of type $< \alpha$.

In the following statements it is to be understood that Y is a closed subset of X—a restriction which will later be removed.

(3) If Y is of type α, then every closed subset Z of Y is of type $\leqq \alpha$.

For $Y = Y_1 \cup \cdots \cup Y_n$, where Y_i is irreducibly of type $\alpha_i \leqq \alpha$. Then $Y_i \cap Z$ is of type $\leqq \alpha_i$, by (2), so Z is of type $\leqq \alpha$, by (1).

(4) *Y is of type α if and only if $Y = F_1 \cup \cdots \cup F_n$, where F_i is closed and irreducibly of type α_i and* $\max(\alpha_1, \cdots, \alpha_n) = \alpha$.

If Y is expressible in this form, Y has type $\leqq \alpha$ by (1); but if Y is of type $< \alpha$, then each $\alpha_i < \alpha$ by (3), and therefore $\max(\alpha_1, \cdots, \alpha_n) < \alpha$, which is impossible. Conversely, if Y is of type α, the definition shows that $Y = F_1 \cup \cdots \cup F_n$, where F_i is closed and irreducibly of type $\leqq \alpha$, say of

[5] A space Y is "trivial" if its only closed subsets are $\emptyset$ and Y.

type α_i. Let $\beta = \max(\alpha_1, \cdots, \alpha_n)$; thus $\beta \leqq \alpha$. But (1) shows that Y has type $\leqq \beta$; hence $\beta = \alpha$.

(5) *Y is irreducibly of type α, if and only if Y is irreducible and of type α.*

If Y is irreducible and of type α, we express Y as in (4) with n as small as possible. Because Y is irreducible, $n = 1$, and then $Y = F_1$, irreducibly of type $\alpha_1 = \alpha$. Conversely, if Y is irreducibly of type α, suppose $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are proper closed subsets of Y; by (2), Y_1 and Y_2 have types $< \alpha$, and (1) gives a contradiction.

(6) *If Y is of type α, and $\beta < \alpha$, then Y has a closed subset Z of type β.*[6]

As Y is a closed subset of X which is not of type $< \beta$, Y contains a minimal closed subset Z (of X) with this property; and Z is irreducibly of type β, by definition.

(7) *The type of a closed subset Y of X does not depend on the containing space X.*

It is enough to show that if Y has type α in X, Y has type α when the containing space is Y. We prove by transfinite induction on β that if a closed subset Z of Y has type β in X, it has type β in Y, and conversely. For $\beta = -1$ this is clear. Assume it true for all $\beta < \gamma$, where $\gamma \leqq \alpha$. If Z is *irreducibly* of type γ in X, then Z is a minimal closed subset of X which is not of type $< \gamma$ in X; in view of the induction hypothesis, it is also a minimal closed subset of Y which is not of type $< \gamma$ in Y, and so it is (irreducibly) of type γ in Y. If Z is of type γ in X but not necessarily irreducible, if follows from (4) and the preceding that Z is of type γ in Y. The converse is established by substantially the same argument.

We can thus speak of the *type* of a hereditarily compact space Y, independent of the containing space X; it is, of course, a topological invariant of Y. It follows from (7) that, in propositions (2)-(6), the hypothesis that Y is closed in X can be omitted; these propositions apply to arbitrary hereditarily compact spaces Y.

(8) *There exist hereditarily compact T_1 spaces of type α, for every ordinal α.*

To construct one such space X, let A denote the section of ordinals $< \alpha$, beginning with -1 (that is, we count -1 as an ordinal), let I denote any

[6] It follows that, if α is finite, Y has a family of non-empty closed proper irreducible subsets, well-ordered under inclusion and of ordinal α. This need not be true when α is infinite.

infinite set, and put $X = A \times I$. The closed sets in X are defined to be those of the form $(B \times I) \cup F$, where B is an arbitrary section of A (or A itself) and F is an arbitrary finite set. It is easily verified (using Theorem 1(5)) that X is a hereditarily compact T_1 space; and a straightforward transfinite induction on α shows that X is irreducibly of type α.

The hereditarily compact T_1 spaces of type 0 are finite unions of 1-point spaces—that is, they are the finite (non-empty) T_1 spaces. The hereditarily compact T_1 spaces X irreducibly of type 1 are those of the following form: X is an infinite set and its closed subsets are just X and its finite subsets. The hereditarily compact spaces of finite type n are those of "dimension n" in the sense: $n+1$ is the greatest length of any strictly decreasing sequence of irreducible closed non-empty subsets. This agrees with the usual dimension for algebraic varieties in the Zariski topology. For $n > 1$, and still more for infinite types, there are surprisingly many of them; we return to this in § 7. In the next section we show how all hereditarily compact T_1 spaces of type α can be "constructed" if we know enough about those of types $< \alpha$.

5. Dual direct systems. Let $\{F_\lambda, f_\lambda^\mu\}$ be a direct system of spaces F_λ (the suffixes λ running over a directed set Λ) and maps $f_\lambda^\mu : F_\lambda \to F_\mu$ $(\lambda < \mu)$ subject to the usual rule $f_\mu^\nu f_\lambda^\mu = f_\lambda^\nu$ for $\lambda < \mu < \nu$. We assume further that the maps f_λ^μ are *closed*. Let S be the limit space; thus a point of S is an equivalence class $\{x\}$ of representatives $x = \{x_\lambda\}$, where $x_\lambda \in F_\lambda$ for $\lambda > \lambda(x)$ and $f_\lambda^\mu(x_\lambda) = x_\mu$ for $\mu > \lambda > \lambda(x)$; two representatives $\{x_\lambda\}$ and $\{y_\lambda\}$ are equivalent if and only if $x_\lambda = y_\lambda$ for $\lambda > \lambda(x, y)$. We give S, *not* the usual direct limit topology, but one which (roughly speaking) uses closed sets instead of open sets. Let f_λ be the usual mapping of F_λ in S, defined as follows: given $x_\lambda \in F_\lambda$ write $x_\mu = f_\lambda^\mu(x_\lambda)$ for $\mu > \lambda$, and put

$$f_\lambda(x_\lambda) = \{\{x_\mu \mid \mu > \lambda\}\} \in S.$$

The closed sets of S are to be the intersections of sets of the form $f_\lambda(K_\lambda)$, where K_λ is closed in F_λ; that is, the sets $S - f_\lambda(K_\lambda)$ form a basis of open sets. It is easily verified that, if $\lambda < \nu$, $f_\nu f_\lambda^\nu = f_\lambda$, and thence that, if $\nu > \lambda, \mu$, $f_\lambda(K_\lambda) \cup f_\mu(K_\mu) = f_\nu(f_\lambda^\nu(K_\lambda) \cup f_\mu^\nu(K_\mu)) = f_\nu(K_\nu)$, where K_ν is closed in F if K_λ, K_μ are closed in F_λ, F_μ. Hence this does define a topology on S, the coarsest in which each f_λ is closed. (In general, the mappings f_λ will not be continuous, even if each f_λ^μ is continuous.) We call S, with this topology, the "dual direct limit space" of the system $\{F_\lambda, f_\lambda^\mu\}$. Clearly S is a T_1 space if each F_λ is T_1.

We are particularly concerned with the case in which each f_λ^μ is 1-1 and

continuous (and thus a homeomorphism into); we then call $\{F_\lambda, f_\lambda^\mu\}$ an *imbedding system*. In this case the closed proper subsets of S are simply the sets $f_\lambda(K_\lambda)$, where K_λ is closed in $F_\lambda (\lambda \in \Lambda)$, and each f_λ is a homeomorphism into.

THEOREM 6. *The dual direct limit space S of an imbedding system $\{F_\lambda, f_\lambda^\mu\}$ of hereditarily compact spaces, each of type $< \alpha$, is hereditarily compact and of type $\leqq \alpha$; it is irreducible providing no f_λ^μ is onto, and T_1 if each F_λ is. Conversely, every irreducible hereditarily compact T_1 space of type $\alpha > 0$ is homeomorphic to the dual direct limit space of an imbedding system $\{F_\lambda, f_\lambda^\mu\}$, where each F_λ is hereditarily compact, T_1, and of type $< \alpha$, and no f_λ^μ is onto.*

In a sense, this theorem determines all hereditarily compact T_1 spaces by transfinite induction over the type; for any such space is a finite union of closed irreducible subsets of no greater type (4(4) and 4(5)).

Proof. Suppose each F_λ is hereditarily compact and of type $< \alpha$. If there could be an infinite strictly decreasing sequence of closed proper subsets $f_{\lambda_n}(K_{\lambda_n})$ of S $(n = 1, 2, \cdots)$, where K_{λ_n} is closed in F_{λ_n}, the sets $f_{\lambda_1}^{-1}(f_{\lambda_n}(K_{\lambda_n}))$ would form a strictly decreasing sequence of closed subsets of F_{λ_1}, which is impossible (Theorem 1(5)). Hence S is hereditarily compact. Each proper closed subset of S, being homeomorphic to a closed subspace of some F_λ, is of type $< \alpha$ (by 4(3) and 4(7)); hence S is of type $\leqq \alpha$. If S is reducible, it is the union of two sets of the form $f_\lambda(K_\lambda)$, $f_\mu(K_\mu)$. Take $\nu > \lambda, \mu$: it follows that $f_\nu(F_\nu) = S$, and thence (because the mappings are 1-1) that f_ν^ρ is onto for all $\rho > \nu$. The T_1 property is obvious.

(Conversely, if the direct limit space S of an imbedding system $\{F_\lambda, f_\lambda^\mu\}$ is hereditarily compact and of type $\leqq \alpha$, or is T_1, then the same is true of each F_λ; for F_λ is homeomorphic to a closed subspace of S.)

If X is irreducible, T_1, and hereditarily compact of type $\alpha > 0$, let $\{F_\lambda\}$ be the family of its closed proper subsets, ordered by (proper) inclusion; as $F_\lambda \cup F_\mu$ is also a closed *proper* subset, the family is directed.[7] Let f_λ^μ be the inclusion map ("identity") for $F_\lambda \to F_\mu$. This defines an imbedding system; let S be its dual direct limit space. It is easily verified that S is homeomorphic to X (the T_1 axiom guarantees that the obvious map of S in X is onto), and that the other properties asserted hold good.

Remark. In the first part of Theorem 6, to ensure the hereditary com-

[7] The T_1 axiom is used here to produce a closed proper subset of X properly containing F_λ and F_μ when $\lambda = \mu$.

pactness of the direct limit S of a direct system of hereditarily compact spaces, we have assumed that each f_λ^μ is closed, continuous and 1-1. None of these assumptions can be omitted; nor can the usual (instead of the dual) direct limit topology be used.

6. Standard operations and types.

LEMMA 1. *If every proper closed subset Z of a space X is hereditarily compact and of type $< \alpha$, then X is hereditarily compact and of type $\leqq \alpha$.*

That X is hereditarily compact follows from Theorem 1(5); the rest follows from the way in which types were defined.

THEOREM 7. *If Y is any subspace of a hereditarily compact space X of type α, then Y is hereditarily compact and of type $\leqq \alpha$.*

This is proved by transfinite induction on α. We may assume that the theorem is true for all smaller types, and also (since we may replace X by $\bar{Y}$, in view of 4(3)) that $X = \bar{Y}$. Suppose first that Y is irreducible; then X is also irreducible (3(1)). Any proper relatively closed subset of Y is of the form $Y \cap Z$, where Z is a closed proper subset of X; say Z has type β. Then $\beta < \alpha$ because X is irreducible; the hypothesis of induction then gives that the type of $Y \cap Z$ is $\leqq \beta < \alpha$, and by Lemma 1 the type of Y is $\leqq \alpha$. Finally, if Y is not irreducible, we have $Y = Y_1 \cup \cdots \cup Y_n$, where each Y_i is (relatively) closed and irreducible and of type α_i say. By the result just established, $\alpha_i \leqq \alpha$ $(i = 1, \cdots, n)$; thus the type of $Y = \max(\alpha_1, \cdots, \alpha_n) \leqq \alpha$.

THEOREM 8. *The union X of a finite number of hereditarily compact spaces $Y_1, \cdots, Y_n$ is hereditarily compact; and if Y_i is of type α_i and X of type α, then*

$$\max(\alpha_1, \cdots, \alpha_n) \leqq \alpha \leqq 1 + \alpha \leqq (\Sigma)(1 + \alpha_i).$$

Here $(\Sigma)\alpha_i$ denotes the "natural" sum of the ordinals $\alpha_1, \cdots, \alpha_n$; that is, we express each α_i in the form $\omega_0^{\xi_1} k_{i1} + \omega_0^{\xi_2} k_{i2} + \cdots + \omega_0^{\xi_m} k_{im}$, where the ordinals ξ_j satisfy $\xi_1 > \xi_2 > \cdots > \xi_m = 0$, and $k_{i1}, \cdots, k_{im}$ are positive integers or 0, and define

$$(\Sigma)\alpha_i = \omega_0^{\xi_1} \sum k_{i1} + \cdots + \omega_0^{\xi_m} \sum k_{im}.$$

(See [11, pp. 363, 364].) When $\alpha_1, \cdots, \alpha_n$ are all finite, this coincides with their ordinary sum.

Note that $1 + \alpha = \alpha + 1$ if α is finite, but $1 + \alpha = \alpha$ otherwise.

Proof. That X is hereditarily compact is obvious, and that $\alpha \geqq \max(\alpha_1, \cdots, \alpha_n)$ follows from Theorem 7. To prove the remaining inequality, we use transfinite induction over the ordered n-ples $(\alpha_1, \cdots, \alpha_n)$ of ordinal numbers (each $\leqq$ some large enough α^*), ordered lexicographically (n being fixed); this is a well-ordered family. It is convenient to count -1 as an ordinal here. Thus the induction starts with each $\alpha_i = -1$; each Y_i is empty, so X is empty and of type $\alpha = -1$ as required. Now suppose that the assertion is true for all $(\alpha_1', \cdots, \alpha_n') < (\alpha_1, \cdots, \alpha_n)$. We first assume that each Y_i is *irreducibly* of type α_i. If Z is any proper closed subset of $X = \bigcup Y_i$, then $Y_i \cap Z$ is for at least one i a proper closed subset of Y_i; hence if $Y_i \cap Z$ has type β_i, we have $\beta_i \leqq \alpha_i$ $(1 \leqq i \leqq n)$, and $\beta_j < \alpha_j$ for at least one value of j. Thus $(\beta_1, \cdots, \beta_n) < (\alpha_1, \cdots, \alpha_n)$, and it follows from the induction hypothesis that the type β of Z satisfies

$$1 + \beta \leqq (\sum)(1 + \beta_i) < (\sum)(1 + \alpha_i).$$

Hence, by Lemma 1, the type α of X satisfies $1 + \alpha \leqq (\sum)(1 + \alpha_i)$.

In the general case, let $Y_i = \bigcup \{Y_{ij} \mid j = 1, 2, \cdots, m(i)\}$, where Y_{ij} is relatively closed and irreducible, and for each of the $m(1)m(2)\cdots m(n)$ choices λ of suffixes, put $Z_\lambda = \bigcap_i \bar{Y}_{i\lambda(i)}$. Then $Z_\lambda \cap Y_i \subset Y_{i\lambda(i)}$, so we have $Z_\lambda \subset \bigcup \{Y_{i\lambda(i)} \mid i = 1, 2, \cdots, n\}$. By Theorem 7 and the case already dealt with, the type γ_λ of Z_λ satisfies

$$(1 + \gamma_\lambda) \leqq (\sum)(1 + \text{type of } Y_{i\lambda(i)}) \leqq (\sum)(1 + \alpha_i).$$

But $X = \bigcup Z_\lambda$, a finite union of *closed* sets; hence the type α of X satisfies $\alpha = \max(\gamma_\lambda)$, and the desired relation $(1 + \alpha) \leqq (\sum)(1 + \alpha_i)$ follows.

Remark. The inequalities in (2) are "best possible," even for T_1 spaces, as can be seen by taking X to be the example constructed to prove 4(8).

THEOREM 9. *The product X of a finite number of hereditarily compact spaces $Y_1, \cdots, Y_n$ is hereditarily compact; and if Y_i is of type α_i and no Y_i is empty, then the type of X is $(\sum)\alpha_i$ $(1 \leqq i \leqq n)$.*

It will suffice to prove this when $n = 2$, as then the general result follows by induction over n. As in the proof of Theorem 8 we use transfinite induction over the ordered pairs (α_1, α_2) in lexicographic ordering, and may assume the theorem for products of spaces of types β_1 and β_2 whenever $(\beta_1, \beta_2) < (\alpha_1, \alpha_2)$. Again, as in Theorem 8, we can easily reduce the proof to the case in which Y_1 and Y_2 are irreducible. It readily follows that X is irreducible too. Let Z be any proper closed subset of X; then $X - Z$ contains a set of the form

$U_1 \times U_2$, where U_1, U_2 are non-empty open subsets of Y_1, Y_2. Then $Y_1 - U_1$ and $Y_2 - U_2$ are of types (say) β_1 and β_2, where $\beta_1 < \alpha_1$ and $\beta_2 < \alpha_2$. Hence, by the induction hypothesis, $(Y_1 - U_1) \times Y_2$ and $Y_1 \times (Y_2 - U_2)$ are hereditarily compact and of types $\beta_1(+)\alpha_2$, $\alpha_1(+)\beta_2$. As they are closed in X, Theorem 8 and 4(1) show that their union T is hereditarily compact and of type $\max(\beta_1(+)\alpha_2, \alpha_1(+)\beta_2) < \alpha_1(+)\alpha_2$. But $Z \subset T$, so the same is true of Z; and from Lemma 1 it follows that X is hereditarily compact and of type $\leqq \alpha_1(+)\alpha_2$. To obtain the reverse inequality, suppose (say) $\alpha_2 \neq 0$. For every ordinal $\gamma_2 < \alpha_2$, Y_2 contains a closed proper subset of type γ_2; applying the induction hypothesis again shows that X contains a closed proper subset of type $\alpha_1(+)\gamma_2$. Thus the type of X is greater than $\alpha_1(+)\gamma_2$ for every $\gamma_2 < \alpha_2$, and so is $\geqq \alpha_1(+)\alpha_2$.

If $\alpha_1 = \alpha_2 = 0$ (i. e., to start the induction), Y_1 and Y_2 are trivial spaces; consequently X is trivial too, and so is hereditarily compact and of type 0.

Remark. A product of infinitely many non-trivial spaces is *never* hereditarily compact. For it contains a subspace homeomorphic to $\prod Y_n$ $(n = 1, 2, \cdots)$, where Y_n consists of two points a_n, b_n and (b_n) is open in Y. But this contains the infinite discrete subset $(b_1, a_2, \cdots, a_n, \cdots)$, $(a_1, b_2, a_3, \cdots)$, etc.

THEOREM 10. *Let f be a continuous mapping of a hereditarily compact space X of type α. Then $f(X)$ is hereditarily compact and of type less than $\omega_0^{\alpha+1}$.*

Let $Y = f(X)$. Each $Z \subset Y$ is compact, being a continuous image of the compact set $f^{-1}(Z) \subset X$. To prove the remainder of the assertion, suppose first that X is irreducible. We show by transfinite induction over α that Y has type $\leqq \omega_0^{\alpha}$, if $\alpha \geqq 0$. When $\alpha = 0$, X is trivial; hence Y is trivial, so its type $= 0 < \omega_0^0$. In general, if Z is any closed proper subset of Y, which we may assume to be non-empty, let $f^{-1}(Z) = S_1 \cup \cdots \cup S_n$, where each S_i is a non-empty closed irreducible subset of X, necessarily proper. Let S_i have type β_i, and put $\beta = \max(\beta_1, \cdots, \beta_n)$; thus $\beta < \alpha$, because X is irreducible. By the hypothesis of induction, the type of $f(S_i)$ is $\leqq \omega_0^{\beta}$; and by Theorem 8 the type of $Z = \bigcup f(S_i)$ is $\leqq \omega_0^{\beta} n < \omega_0^{\beta+1} \leqq \omega_0^{\alpha}$. Hence, by Lemma 1, Y has type $\leqq \omega_0^{\alpha}$.

In the general case, we have $X = X_1 \cup \cdots \cup X_m$, where X_j is irreducible of type α_j, and $\alpha = \max(\alpha_1, \cdots, \alpha_m)$ (4(4) and 4(5)). By Theorem 8 and the foregoing, the type of Y is $\leqq \omega_0^{\alpha} m < \omega_0^{\alpha+1}$.

Remark. The bound for the type of $f(X)$ here is sharp, even for T_0

spaces, and even if f is 1-1. But if α is finite and $f(X)$ is T_1, its type is $< \omega_0^{\alpha}$, which is now "best possible"; for infinite α, Theorem 10 is sharp even for T_1 spaces and 1-1 mappings. However, if f is *closed* and continuous, it is easily seen that the type of $f(X)$ does not exceed the type of X.

7. Countable spaces. The simplest hereditarily compact spaces are those which have at most countably many closed (or open) sets. Concerning these we have:

THEOREM 11. *Let X be a hereditarily compact space. Then:*

(1) *The family of open subsets of X is countable if and only if X has a countable base.*

(2) *If X is T_0 and has a countable base, then X is countable.*

(3) *If X is T_1, X has a countable base if and only if it satisfies the first axiom of countability.*

Proof. (1) Let $B_1, B_2, \cdots$ be a countable base of open sets. We show that the open sets coincide with the *finite* unions of the sets B_i—which evidently form a countable family. In fact, if U is open, U is a union of sets B_i, and being compact is covered by a finite number of them. "Only if" is trivial.

(2) By (1), X has at most $\aleph_0$ distinct closed sets; but the sets $\bar{x}$ $(x \in X)$ are all distinct.

(3) Assuming X is "first countable," we first show that X is countable, using transfinite induction over the type α of X. We may clearly assume that X is irreducible and not empty. Pick $p \in X$ and let $U_1, U_2, \cdots$ be a basis of open neighborhoods of p; thus $\bigcap U_n = (p)$. Put $F_n = X - U_n$; then F_n has type $< \alpha$, so by the induction hypothesis F_n is countable. Hence X is countable.[8] If the points of X are enumerated as $q_1, q_2, \cdots$, and if $V_{n1}, V_{n2}, \cdots$ is a basis of open neighborhoods of q_n, the sets V_{nm} evidently form a countable basis for X.

The converse implication is trivial.

Remark. There are hereditarily compact T_0 spaces which satisfy the first axiom of countability and have arbitrarily large cardinal, and there are hereditarily compact T_1 spaces (of type 1) which are separable but have arbitrarily large cardinal.

---

[8] This argument applies, more generally, if instead of assuming that X is T_1 and first countable, we assume that each point of X is a G_δ in X.

One might expect that, conversely, a countable T_1 hereditarily compact space has to satisfy the first axiom of countability, at least at one point, especially in view of a theorem of S. Mrówka [8] asserting that a compact T_2 space with fewer than $2^{\aleph_1}$ points must satisfy the first axiom of countability at some point. But this is not the case, as the following example shows.

Example 1. *There exists a countable hereditarily compact T_1 space of type 2, having c closed subsets, and not having a countable base of neighborhoods at any point.*

The example requires the following lemma, which is due to Sierpinski (cf. [11, p. 77]).

LEMMA 2. *Let S be a set with $\aleph_0$ elements. There is a family of c distinct infinite subsets A_x of S, every two of which intersect in at most a finite set.*

We may take S to be the set of rational numbers, and for each real number x take A_x to be a sequence of rational numbers converging to x.[9]

Now topologize S by taking its closed sets to be: S, and all sets of the form $E \cup \bigcup A_{x_i}$ $(i=1,2,\cdots,n)$, where n is a non-negative integer and E is finite. S is easily seen to be irreducible, hereditarily compact and T_1. The closed sets of type 0 are the non-empty sets E; the irreducible closed sets of type 1 are the sets A_x; and thus S is of type 2. We may assume that, given $p \in S$, there are uncountably many sets A_x which do not contain p; for the set of points p for which this is not true must be finite, and we simply omit them from S. If $V_1, V_2, \cdots$ is a countable basis of open neighborhoods of p, we have $V_m = S - (E_m \cup \bigcup \{A_x \mid x \in F_m\})$, where E_m, F_m are finite. As $\bigcup F_m$ is countable, there exists a suffix $y \notin F_m$ for which $p \notin A_y$, and $S - A_y$ is a neighborhhod of p. It must contain some V_m and then $A_y \subset E_m \cup \bigcup \{A_x \mid x \in F_m\}$. But each $A_y \cap A_x$ $(x \in F_m)$ is finite, so A_y is finite, giving a contradiction as required.

A *countable* hereditarily compact space X can evidently have at most c closed subsets; its type must therefore have cardinal $\leqq c$ (from 4(6)). Further, as there are at most 2^c ways of selecting the c sets which are to be closed in X, there can be at most 2^c nonhomeomorphic countable hereditarily compact (or, indeed, countable) spaces. We show now that these trivial estimates are in fact "best possible," even for T_1 spaces. That there can be as many as c closed sets has been shown by Example 1.

[9] This simple proof of Lemma 2 is also due to Sierpinski.

Example 2. *There exists, for each ordinal* λ *of cardinal* $\leqq c$, *a countable hereditarily compact* T_1 *space of type* λ.

We use transfinite induction over λ. Using the sets S, A_x of Lemma 2, and noting that the number of ordinals $\beta < \lambda$ is at most c, we assign to each $\beta < \lambda$ one or more spaces A_x and topologise them as (countable) hereditarily compact T_1 spaces of type β. Now define a topology on S by taking the closed sets to be: S, and all sets of the form $\bigcup F_{x_i}$ $(i = 1, 2, \cdots, n)$, where F_{x_i} is closed in A_{x_i}. One easily verifies that this does give a topology on S, in which S is T_1 and hereditarily compact, and that the subspace topology it induces on each A_x coincides with the topology originally assigned to A_x. Hence S is irreducibly of type λ.

THEOREM 12. *There are* 2^c *nonhomeomorphic countable hereditarily compact* T_1 *spaces.*

Let Ω denote the smallest ordinal of cardinal c; let P be the set of ordinals less than Ω, Q the set of non-limit ordinals in P, and R any subset of Q. Thus there are 2^c distinct sets R, and for each of them there are c elements in $P - R$. We construct for each R a corresponding space as follows. Again we use Lemma 2. Let $x \leftrightarrow \alpha(x)$ be a 1-1 correspondence between the set of suffixes x and the set $P - R$. From Example 2, we can give each set A_x a hereditarily compact T_1 topology, irreducibly of type $\alpha(x)$. As in Example 2, we can extend these topologies to a hereditarily compact T_1 topology on S. Now the sets A_x will be precisely the maximal proper irreducible closed subsets of S. Hence the topology of S determines the family of types of the sets A_x, and hence determines R. That is, different sets R give nonhomeomorphic spaces S, and the theorem follows.

It would be interesting to know how many nonhomeomorphic countable hereditarily compact T_1 spaces have a given type α. By a slight modification of the above argument one can show that this number is at least $2^{|\alpha|}$ if $\aleph_0 \leqq |\alpha| \leqq c$, where $|\alpha|$ denotes the cardinal of α.

It would also be interesting to have corresponding estimates for hereditarily compact T_1 spaces of larger cardinals. The above methods can of course be extended, but do not suffice to settle the questions in general; the difficulty is that the analogue of Lemma 2 is false for "most" cardinals (see [12]).

THE UNIVERSITY,
MANCHESTER, ENGLAND.

REFERENCES.

[1] R. W. Bagley, E. H. Connell and J. D. McKnight, Jr., "On properties characterizing pseudo-compact spaces," *Proceedings of the American Mathematical Society*, vol. 9 (1958), pp. 500-506.

[2] E. Hewitt, "Rings of real-valued continuous functions I," *Transactions of the American Mathematical Society*, vol. 64 (1948), pp. 45-99.

[3] K. Iseki, "On weakly compact and countably compact topological spaces," *Proceedings of the Japan Academy*, vol. 33 (1957), pp. 323-324.

[4] ———, "*AU*-covering theorem and compactness," *ibid.*, pp. 363-367.

[5] S. Kasahara, "On weakly compact regular spaces II," *ibid.*, pp. 255-259.

[6] ———, "A note on some topological spaces," *ibid.*, pp. 453-454.

[7] S. Mardešić and P. Papić, "Sur les espaces dont toute transformation réelle continue est bornée," *Glasnik Matematičko-Fizički Astronomski*, Serija II, vol. 10 (1955), pp. 225-232.

[8] S. Mrówka, "On the potency of compact spaces and the first axiom of countability," *Bulletin de l'Académie Polonaise des Sciences* Sér. Sci. Math. Astr. Phys. 61 (1958), pp. 7-9.

[9] L. Nollet, "A propos de la locale compacité des espaces topologiques," *Bulletin de la Société Royale des Sciences de Liège*, vol. 28 (1959), pp. 5-13.

[10] J. P. Serre, "Faisceaux algébriques cohérents," *Annals of Mathematics* (2), vol. 61 (1955), pp. 197-278.

[11] W. Sierpinski, *Cardinal and ordinal numbers*, Warsaw, 1958.

[12] A. Tarski, "Sur la décomposition des ensembles en sous-ensembles presques disjoints," *Fundamenta Mathematicae*, vol. 12 (1928), pp. 186-205.

IMBEDDING OF HOLOMORPHICALLY COMPLETE COMPLEX SPACES.*

By RAGHAVAN NARASIMHAN.

1. Introduction.

(1.1) In a previous paper [6], the author proved that any open Riemann surface has a closed, nonsingular imbedding in the complex number space $\boldsymbol{C}^3$. It is our object, in this paper, to obtain analogous theorems on the imbedding of higher dimensional holomorphically complete complex spaces. (Our terminology is that of [4], [5]; all complex spaces considered are "β-Räume" in the sense of [5].) We prove three theorems which contain the following statements:

1. Any Stein manifold of dimension n admits a one-one, proper, regular holomorphic map in $\boldsymbol{C}^{2n+1}$.

2. Any holomorphically complete complex space of dimension n admits a one-one, proper, holomorphic map in $\boldsymbol{C}^{2n+1}$ which is regular at every uniformizable point.

3. If locally the holomorphically complete space X can be realized as an analytic set in an open set in $\boldsymbol{C}^N$ $(N > n$ fixed) with the induced structure from $\boldsymbol{C}^N$, then there is a one-one proper map ϕ of X in $\boldsymbol{C}^{N+n}$ whose image with the induced structure from $\boldsymbol{C}^{N+n}$ is isomorphic to X by means of ϕ.

In all cases, these imbeddings are dense in the space of all holomorphic mappings into the corresponding number spaces, with the compact convergence topology.

These results have the effect of refining the theorem announced by R. Remmert [7] that any holomorphically complete space has a one-one proper map into $\boldsymbol{C}^m$ for some m. They also improve a sharper version of the theorem of Remmert which Professor H. Grauert demonstrated to the author (in a lecture at the Tata Institute of Fundamental Research, Bombay), as he said, by the method of Remmert. Our proofs are based on this version given by Professor Grauert. Since no proof of even the original theorem of Remmert has so far appeared, the proof presented by Professor Grauert is given in § 2

* Received May 10, 1960.

(see Theorem 1). It is an important step in the proof of the main theorems of this paper.

The author takes great pleasure in recording here his indebtedness to Professor H. Cartan for many valuable suggestions and discussions, which led to the formulation and proof of the theorems given here. In particular, the proof of Lemma 1 is due to Professor Cartan, and replaces a more complicated proof of the author.

(1.2) *Some preliminary remarks.* The problem that we consider is that of closed imbeddings of complex spaces in number spaces. By this we mean the existence of a one-one proper holomorphic map into $\boldsymbol{C}^N$. We remark that a *proper map* is one for which the inverse image of a compact set is compact, and that a *regular map* of a complex manifold X is one for which the Jacobian matrix has at any $x_0 \in X$ a rank equal to the dimension of X at x_0.

For a closed imbedding of a complex space X to exist, the following conditions must hold.

i) X is holomorph-convex, i.e. for any infinite discrete set on X, there exists a holomorphic function on X which is unbounded on this set.

ii) Given two distinct points of X, there is a holomorphic function on X taking distinct values at these two points.

iii) X is countable at infinity.

If these conditions are satisfied, we say that X is *holomorphically complete* or that it is a *Stein space.* The properties which we require of such spaces are proved in [4]. We remark that, if X is connected, i) implies iii) (see [4]).

Condition i) is equivalent to the following: for any compact set $K \subset X$, the set

$$\hat{K} = \{x \in X \mid |f(x)| \leqq \sup_{y \in K} |f(y)| \text{ for all } f \text{ holomorphic on } X\}$$

is also compact. We shall denote throughout the paper, the space of holomorphic functions on X and the space of holomorphic mappings of X into $\boldsymbol{C}^N$ by $\mathcal{H}$ and $\mathcal{H}^N$ respectively. They are provided with the topology of compact convergence. If $f \in \mathcal{H}^N$, $f = (f_1, \cdots, f_N)$, we shall set $|f| = \max_i |f_i|$.

It is not always possible to find a proper holomorphic map of a Stein space of dimension n into $\boldsymbol{C}^n$. To prove this, let X be an irreducible K-complete space admitting non-constant bounded holomorphic functions (e.g. bounded domains in $\boldsymbol{C}^n$). Suppose that $\phi: X \to \boldsymbol{C}^n$ is a proper holomorphic

map. Then, if $z \in \boldsymbol{C}^n$, $\phi^{-1}(z)$ is a compact analytic set and so is a finite set (by the maximum principle). Hence, by [9, Satz 23], $\phi(X)$ is an analytic set of dimension n in $\boldsymbol{C}^n$ so that $\phi(X) = \boldsymbol{C}^n$. Thus $X \to \boldsymbol{C}^n$ is a ramified covering. This implies that any holomorphic function f on X satisfies an equation

$$f^m + a_1(z) f^{m-1} + \cdots + a_m(z) = 0,$$

where $a_1(z), \cdots, a_m(z)$ are holomorphic on $\boldsymbol{C}^n$ (the $a_i(z)$ are the elementary symmetric functions of the values of f at the points lying over z). If f is bounded on X, the $a_i(z)$ are bounded on $\boldsymbol{C}^n$ and so are constant; this means that f itself is constant. This cannot be the case since X has non-constant holomorphic functions on it.

Let $U \subset X$ be an open subset of a Stein space X. We say that U *is X-convex* if the following holds:

Let $K \subset U$ be a compact set. Let

$$K' = \{x \in U \mid |f(x)| \leqq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathfrak{H}\}$$

($\mathfrak{H}$ being the space of holomorphic functions on X). Then K' is compact. This is equivalent to saying that the set $\hat{K} \cap U$ is compact.

It is a well known theorem (usually called the Oka-Weil theorem) that if U is X-convex, then any holomorphic function on U can be approximated, uniformly on compact subsets of U, by functions holomorphic on X. This result implies that an open set U is X-convex if and only if U is a Stein space and any holomorphic function on U can be approximated, uniformly on compact sets of U, by holomorphic functions on X.

We make another remark. If K is a compact set of X, then any connected component of $\hat{K}$ meets K. To see this, let A be a connected component of $\hat{K}$ not meeting K. Let V_1, V_2 be disjoint open sets in X, $V_1 \cup V_2 \supset \hat{K}$, $V_1 \supset A$, $V_2 \supset K$. Let W be an X-convex open set of X containing $\hat{K}$ and contained in $V_1 \cup V_2$. Let f be holomorphic on X, $|f(x) - 1| < \frac{1}{2}$ for $x \in A$, $|f(x)| < \frac{1}{2}$ for $x \in K$. Then clearly $|f(x)| > |f(y)|$ if $x \in A$, $y \in K$ so that $A \subset \hat{K}$; contradiction. This implies that if U is an open set such that $\hat{K} \cap U$ is compact, where K is a compact subset of U, then $\hat{K} \subset U$. In particular, if U is X-convex and $K \subset U$, then $\hat{K} \subset U$.

The following result is an easy consequence of the definition of X-convexity:

(1.3) If X_1 and X_2 are Stein spaces, $f: X_1 \to X_2$ a holomorphic map, then the inverse image of an X_2-convex open set is X_1-convex.

We shall later use this.

2. Decomposition into admissible systems. In this section we shall prove, following H. Grauert, that any Stein space of dimension n can be split into $2n+1$ systems U^λ of relatively compact disjoint open sets $U^\lambda{}_i$ such that the following approximation theorem holds: if f_i is a holomorphic function on $U^\lambda{}_i$, and K_i is a compact set of $U^\lambda{}_i$, (for each i) then there is a holomorphic function F on X approximating f_i on K_i for all i. The proof may be looked upon as generalizing the following simple construction for open Riemann surfaces X. Let T_1 be any triangulation of X. Then there is a triangulation T_2 of X such that the sides of T_2 meet those of T_1 only in points. Then $U^\lambda{}_i$ $(\lambda=1,2)$ are the open triangles of T_λ, $U^3{}_i$ are small parametric discs about the intersections of the sides of T_1 with those of T_2.

We begin with a definition.

A *locally finite* system $\{U_i\}$ of *relatively compact* open sets of X is said to be an *admissible system* if

i) $U=\bigcup_i U_i$ is X-convex, $U_i\cap U_j=\emptyset$ if $i\neq j$,

ii) there exists a sequence of open sets $\{B_\nu\}$, $B_\nu \subset\subset B_{\nu+1}$, $\bigcup_\nu B_\nu = X$ such that for each ν, $B_\nu \cup U$ is X-convex.

We introduce certain conventions which we follow throughout. We remark that the sequence B_ν in ii) may be assumed to have the following property also: if $U_i\cap B_\nu\neq\emptyset$, then $U_i\subset B_\nu$. In fact, we have only to replace B_ν by the union B'_ν of B_ν with the U_i that meet it and pass to a subsequence B'_{ν_k} (to ensure that $B'_{\nu_k}\subset\subset B'_{\nu_{k+1}}$). A sequence satisfying this additional condition, we shall, for convenience, call an *associated sequence* of $\{U_i\}$. It is clear that each B_ν of an associated sequence is X-convex.

The theorem given by Professor Grauert then is as follows:

THEOREM 1. *If X is a Stein space of dimension n, then there exist $2n+1$ admissible systems $\{U^\lambda{}_i\}$, $\lambda=1,\cdots,2n+1$, such that*

$$\bigcup_{\lambda=1}^{2n+1}\bigcup_{i=1}^{\infty} U^\lambda{}_i = X.$$

Proof of Theorem 1. We begin the proof by showing that it is sufficient to prove the existence of an admissible system $\{U_i\}$, such that $A=X-U$, $(U=\bigcup_i U_i)$ is a real analytic set on X which does not contain any point of a given countable set on X. To prove this, we remark that it is easy to deduce from [11, § 8] that the following is true. If M_1 is a real analytic set on X there is a countable set $\{x_m\}$ on M_1 such that if M_2 is real analytic

and does not contain any point x_m, then $\dim(M_1 \cap M_2) < \dim M_1$. It follows from this remark that there are $2n+1$ admissible systems $\{U^\lambda_i\}$, $\lambda = 1, \cdots, 2n+1$, so that if $A_\lambda = X - \bigcup_i U^\lambda_i$, then $\dim(A_1 \cap \cdots \cap A_k) \leqq 2n - k$ for $k = 1, \cdots, 2n+1$. Clearly then $\bigcap_1^{2n+1} A_\lambda = \varnothing$ and we obtain the theorem.

We proceed to the proof of the existence of an admissible system $\{U_i\}$ with the properties stated. There exists a sequence of open sets $B_\nu \subset\subset B_{\nu+1}$, $\bigcup_\nu B_\nu = X$ such that

$$(*) \qquad B_\nu = \{x \in B_{\nu+1} \mid |\Re\phi_{\nu i}| < 1, |\Im\phi_{\nu i}| < 1, \phi_{\nu i} \in \mathfrak{H}, i = 1, \cdots, k_\nu\}$$

(so that in particular, B_ν is X-convex). To prove this, let K be a compact set with $K = \hat{K}$, where

$$\hat{K} = \{x \in X \mid |f(x)| \leqq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathfrak{H}\}.$$

Let C be any compact set and suppose that there is an open set U with $K \subset U \subset C$. Since $K = \hat{K}$, if $x \in C - K$, then, there is an $f \in \mathfrak{H}$ with $|f(x)| > 2$, $|f(y)| < 1$ for $y \in K$. From the compactness of $C - U$, it follows that there exist finitely many functions $f_1, \cdots, f_k \in \mathfrak{H}$ such that $\max_i |f_i(x)| > 2$ if $x \in C - U$, < 1 if $x \in K$. If $B = \{x \in C \mid |\Re f_i| < 1, |\Im f_i| < 1\}$, then B is an open set with $\bar{B} \subset U$. The existence of the sequence B_ν satisfying (*) is an easy consequence of this remark.

Let $Q_N(R)$ $(R > 0)$ be the "cube" $\{z \in \boldsymbol{C}^N \mid |\Re z_i| < R, |\Im z_i| < R\}$, $z = (z_1, \cdots, z_N)$. We shall, by convention, suppose that $\boldsymbol{C}^N \subset \boldsymbol{C}^M$ if $M \geqq N$. A "rectangle" in $\boldsymbol{C}^N$ will be a set $\{z \in \boldsymbol{C}^N \mid a_i < \Re z_i < b_i; c_i < \Im z_i < d_i, i = 1, \cdots, N\}$, where $a_i < b_i$, $c_i < d_i$ are real numbers.

Let $m_\nu = k_1 + \cdots + k_\nu$ (see (*)). It is clear that the $\phi_{\nu i}$ give a proper map of B_ν into $Q_{k_\nu}(1)$. We remark that if $1 \leqq \alpha_\nu \leqq 1 + \epsilon_\nu$, then the domains $\{x \in B_{\nu+1} \mid |\Re\alpha_\nu\phi_{\nu i}| < 1, |\Im\alpha_\nu\phi_{\nu i}| < 1\}$ satisfy the condition corresponding to (*) if the ϵ_ν are small enough.

Let $\Phi_1 : B_1 \to Q_{k_1}(1)$ be the mapping defined by $\phi_{11}, \cdots, \phi_{1k_1}$; let $R_2 > \sup_{x \in B_2} |\Phi_1(x)|$ and let $\Phi_2 : B_2 \to Q_{m_2}(R_2)$ be the mapping $(\Phi_1, R_2\phi_{21}, \cdots, R_2\phi_{2k_2})$. Let $R_3 > \sup_{x \in B_3} |\Phi_2(x)|$, $\Phi_3 : B_3 \to Q_{m_3}(R_3)$ the mapping $(\Phi_2, R_3\phi_{31}, \cdots, R_3\phi_{3k_3})$, and so on.

Let $Q_\nu = Q_{m_\nu}(R_\nu)$ and let Q'_ν be the cylinder on Q_ν as base in $Q_{\nu+1}$ (in the obvious sense). It is clear that Φ_ν is a proper map of B_ν in Q_ν; moreover, *because of condition* (*), *the inverse image in* $B_{\nu+1}$ *of the rectangle* Q'_ν *by* $\Phi_{\nu+1}$ *is precisely* B_ν.

Let $U_1 = B_1$. The boundary hyperplanes of Q'_1 in Q_2 (i.e. the hyperplanes $\Re z_i = \pm 1$, $\Im z_i = \pm 1$, $i = 1, \cdots, k_1$) split Q_2 into a finite number of open rectangles. Let $U_1, U_2, \cdots, U_{p_2}$ be their inverse images by Φ_2 in B_2. (We remark that U_1 is an inverse image, being that of Q'_1.) Consider next the rectangle $Q_{m_2}(R_3)$. The boundary hyperplanes of $Q_2 = Q_{m_2}(R_2)$ and the hyperplanes used to split Q_2 above decompose $Q_{m_2}(R_3)$ into open rectangles, the decomposition being the same as the original one on Q_2. The cylinders on these rectangles as bases in $Q_3 = Q_{m_3}(R_3)$ we denote by $\rho, \rho', \cdots$. Again, it follows from condition (*) that the inverse images in B_3 by Φ_3 of those of these rectangles $\rho, \rho', \cdots$ whose bases lie in Q_2 give the U_k, $k \leqq p_2$. Let U_k $(p_2 < k \leqq p_3)$ be the inverse images of those rectangles $\rho, \rho', \cdots$ which do not meet Q'_2. It is clear that for $k > p_2$, $U_k \cap B_2 = \varnothing$. We continue this process. Obviously the system $\{U_k\}$ of open sets that one obtains in this way is locally finite and disjoint. Moreover since for $k \leqq p_\nu$, $U_k \subset B_\nu$, each U_k is relatively compact. Let $A = X - \bigcup_k U_k$. Clearly $A \cap B_\nu$ is the inverse image under Φ_ν of the union of finitely many hyperplanes in Q_ν so that A is a real analytic set. Moreover, it is clear that A is defined by means of equations of the following type: $\Re\phi_{\nu i} = \pm a_{\mu\nu}$ $(\mu \geqq \nu)$ $\Im\phi_{\nu i} = \pm b_{\mu\nu}$ (where the $a_{\mu\nu}, b_{\mu\nu}$ depend only on the R_ν). Now, it is clear, e.g. if $\alpha_\nu \leqq 2$, that we may choose R_ν to satisfy the following condition: If

$$\Phi_\nu = (\phi_{11}, \cdots, \phi_{1k_1}, \cdots, R_\nu\phi_{\nu 1}, \cdots, R_\nu\Phi_{\nu k_\nu}),$$

put

$$\Psi_\nu = (\alpha_1\phi_{11}, \cdots, \alpha_1\phi_{1k_1}, \cdots, \alpha_\nu R_\nu\phi_{\nu 1}, \cdots, \alpha_\nu R_\nu\Phi_{\nu k_\nu});$$

then we have $R_{\nu+1} > \sup_{x \in B_{\nu+1}} |\Psi_\nu(x)|$. Thus, we may use $\alpha_\nu\phi_{\nu i}$ instead of $\phi_{\nu i}$ to define the U_k. In this case, the set A is obviously defined by $\Re\phi_{\nu i} = \pm a_{\mu\nu}/\alpha_\nu$, $\Im\phi_{\nu i} = \pm b_{\mu\nu}/\alpha_\nu$. Clearly, the α_ν, $1 \leqq \alpha_\nu \leqq 1 + \epsilon_\nu$ can be so chosen that none of these countably many equations is satisfied at any one of a countable set of points.

To complete the proof of Theorem 1, it remains therefore only to show that $\{U_k\}$ is an admissible system. We show this by proving the following: if Q is a cube in $\boldsymbol{C}^N$ and G is the complement in Q of finitely many hyperplanes parallel to the coordinate hyperplanes, then G is Q-convex. This is enough; for if this is proved, then one easily sees by means of remark (1.3) at the end of the introduction, that $B_\nu \cup \bigcup_{p_\nu < k \leqq p_{\nu+1}} U_k$ is $B_{\nu+1}$-convex and so, since $B_{\nu+1}$ is X-convex, it is itself X-convex.

Each open rectangle in Q is of course Q-convex. But the union of disjoint Runge domains is not necessarily itself convex, even in the case of open

Riemann surfaces (see [6]). Professor Grauert proved that G is Q-convex by showing that it has a semicontinuous extension to Q. We give, in Lemma 1, a general criterion for the union of disjoint convex sets to be convex. The result required to complete the proof of Theorem 1 is an immediate consequence.

LEMMA 1. *Let X be a Stein space and let X_1, X_2 be disjoint X-convex open sets. Then the following three conditions are equivalent:*

(i) *$X_1 \cup X_2$ is X-convex;*

(ii) *if $K_1 \subset X_1$ and $K_2 \subset X_2$ are compact, there exists a holomorphic function f on X such that $\Re f > 0$, on K_1, $\Re f < 0$ on K_2;*

(iii) *the function which is $+1$ in X_1 and -1 in X_2 can be approximated, uniformly on compact sets of $X_1 \cup X_2$, by holomorphic functions on X.*

Proof. Clearly (i) implies (iii); (iii) implies (i). First we show that (ii) implies (iii). Let $K_i \subset X_i$ be compact and f be as in (ii). Let $C_i = f(K_i)$ in complex z-plane. Then $\Re z > 0$ on C_1 and $\Re z < 0$ on C_2. Since, by the Behnke-Stein theorem [1], the union of these two half-planes is $\mathbf{C}$-convex, there is a holomorphic function g in the plane with $|g(z) - 1| < \epsilon$ for $z \in C_1$, $|g(z) + 1| < \epsilon$ for $z \in C_2$. Let $h = g \circ f$. Then, we have

$$|h(x) - 1| < \epsilon \text{ for } x \in K_1, \quad |h(x) + 1| < \epsilon \text{ for } x \in K_2$$

which proves (iii).

Finally, (iii) implies (i). Let K be a compact subset of $X_1 \cup X_2$. We have to prove that $\hat{K} \cap (X_1 \cup X_2)$ is compact. Let $K_i = K \cap X_i$; K_i is compact. Moreover, since X_i is X-convex $K'_i = \hat{K}_i \cap X_i$ is compact. We shall show that $\hat{K} \cap (X_1 \cup X_2) = K'_1 \cup K'_2$. Let $x_0 \in X_1 \cup X_2$, $x_0 \notin K'_1 \cup K'_2$. Let $x_0 \in X_1$; then, since $x_0 \in X_1$, $\notin K'_1$, there is a holomorphic function g on X with $|g(x_0)| > \sup_{x \in K_1} |g(x)|$. Let h be holomorphic on X and satisfy

$$|h(x) - 1| < \epsilon \text{ on } K_1 \cup \{x_0\}, \quad |h(x)| < \epsilon \text{ on } K_2$$

(which exists by (iii)). Then, if ϵ is small enough, and we put $f(x) = g(x)h(x)$, we clearly have

$$|f(x_0)| > \sup_{x \in K_1 \cup K_2} |f(x)|;$$

this proves (i) and the lemma is proved.

The proof of Theorem 1 is therefore complete.

The following theorem of approximation is an easy consequence of the definition of admissible systems.

THEOREM 2. *Let $\{U_i\}$ be an admissible system on the Stein space X. Let K_i be a compact set of U_i, f_i a holomorphic function in U_i and $\epsilon_i > 0$. Then there is a holomorphic function f on X such that*

$$|f(x) - f_i(x)| < \epsilon_i \text{ for } x \in K_i, \quad i = 1, 2, \cdots.$$

Proof. Let $\{B_\nu\}$ be an associated sequence. Let $U_i \subset B_1$, $i = 1, \cdots, i_1$, $U_i \subset B_{\nu+1} - B_\nu$ for $i_\nu < i \leqq i_{\nu+1}$ $(\nu \geqq 1)$. Let K'_ν be a compact set of B_ν containing $B_{\nu-1}$ and the K_i for $i \leqq i_\nu$. Let $\delta_\nu > 0$, $\sum_{\mu=\nu+1}^{\infty} \delta_\mu < \frac{1}{2} \min_{i \leqq i_\nu} \epsilon_i$. Since U is X-convex, there is an $F_1 \in \mathcal{A}$ with

$$|F_1(x) - f_i(x)| < \tfrac{1}{2}\epsilon_i \text{ on } K_i (i \leqq i_1).$$

Since $B_1 \cup U$ is X-convex, there is F_2 holomorphic on X with $|F_2 - F_1| < \delta_1$ on K'_1, $|F_2 - f_1| < \frac{1}{2}\epsilon_i$ on K_i for $i_1 < i \leqq i_2$ and so on. Since the limit of holomorphic functions, converging uniformly on compact sets, is also holomorphic (see [5]), $f = \lim_{\nu \to \infty} F_\nu$ is holomorphic on X. One sees easily that $|f - f_i| < \epsilon_i$ on K_i.

We remark that if $\{U_i\}$ is a disjoint family of holomorphically convex open sets which satisfy the conclusion of Theorem 2, then $\{U_i\}$ is locally finite. Also, if K_i is a compact set of U_i $(i = 1, 2, \cdots)$ then there is an admissible system $\{V_i\}$ with $K_i \subset V_i \subset U_i$. In this sense, Theorem 2 characterizes admissible systems.

An immediate consequence of Theorems 1 and 2 is the following result:

Any Stein space of dimension n admits a proper holomorphic map into $\boldsymbol{C}^{2n+1}$.

In fact, let $\{U^\lambda{}_i\}$, $\lambda = 1, \cdots, 2n+1$, be admissible systems with $\bigcup_{i,\lambda} U^\lambda{}_i = X$. Since each $U^\lambda{}_i$ is relatively compact, there exist compact sets $K^\lambda{}_i \subset U^\lambda{}_i$, with $\bigcup_{i,\lambda} K^\lambda{}_i = X$. By Theorem 2, there is a holomorphic function f_λ on X with $|f_\lambda(x) - i| < 1$ for $x \in K_i$, $i = 1, 2, \cdots$. Since the set of $x \in X$ with $|f_\lambda(x)| \leqq i$ $(\lambda = 1, 2, \cdots, 2n+1)$ is then contained in $\bigcup_{\lambda=1}^{2n+1} \bigcup_{j=1}^{i+1} K^\lambda{}_j$, the map $(f_1, \cdots, f_{2n+1})$ is proper.

3. A lemma. In this section, we state and prove a lemma which we shall use in the proof of each of the three main theorems.

LEMMA 2. *Let X be a Stein space. Let N be a positive integer. Suppose that to each compact set K of X is given a set $A(K)$ of functions in $\mathcal{H}^N$. Suppose that $A(K)$ is dense in $\mathcal{H}^N$ and that if $K \subset K'$, then $A(K') \subset A(K)$. Suppose moreover that the following condition holds: (α) if $K \subset (K')^0$, $f \in A(K)$, then there is an $\epsilon > 0$ such that if $g \in \mathcal{H}^N$ and $|f-g| < \epsilon$ on K', then we have $g \in A(K)$.*

Let $\{U^\lambda{}_i\}$, $i = 1, 2, \cdots$; $\lambda = 1, \cdots, N$ be N admissible systems, let $K^\lambda{}_i$ be a compact set of $\{U^\lambda{}_i\}$. Let $\epsilon > 0$ and $\epsilon_i > 0$, $i = 1, 2, \cdots$. Let $g = (g_1, \cdots, g_N) \in \mathcal{H}^N$ be given and let C be a compact set of X.

Then, there exists $f = (f_1, \cdots, f_N) \in \mathcal{H}^N$ such that $|f-g| < \epsilon$ on C, $|f_\lambda - g_\lambda| < \epsilon_i$ on $K^\lambda{}_i$; $\lambda = 1, \cdots, N$; $i = 1, 2, \cdots$ which belongs to $A(K)$ for every compact set K of X.

Proof. Let $\{B^\lambda{}_\nu\}$ be an associated sequence of the admissible system $\{U^\lambda{}_i\}$. Let $'K^\lambda{}_\nu$ be a compact subset of $B^\lambda{}_\nu$ containing $B^\lambda{}_{\nu-1}$ and the $K^\lambda{}_i$ in $B^\lambda{}_\nu$. Let $C^\lambda{}_\nu$ be a compact neighbourhood of $'K^\lambda{}_\nu$ in $B^\lambda{}_\nu$. Let $K'_\nu = \bigcap_{\lambda=1}^{N} {}'K^\lambda{}_\nu$, $C_\nu = \bigcap_{\lambda=1}^{N} C^\lambda{}_\nu$. Suppose that $U^\lambda{}_i \subset B_1$ for $i \leqq i^\lambda{}_1$, $U_i \subset B^\lambda{}_{\nu+1} - B^\lambda{}_\nu$ for $i^\lambda{}_\nu < i \leqq i^\lambda{}_{\nu+1}$. Clearly we have $K'_\nu \subset (K'_{\nu+1})^0$, $\bigcup_\nu K'_\nu = X$. Since $A(K') \subset A(K)$ if $K \subset K'$, if $f \in A(K'_\nu)$ for all ν, then $f \in A(K)$ for any compact K. It is clear that we may suppose that $C \subset K'_1$ and that $\epsilon < \epsilon_i$ for $i \leqq \max_\lambda i^\lambda{}_1$.

$A(K'_1)$ is dense in $\mathcal{H}^N$; hence there is $f^1 = (f^1{}_1, \cdots, f^1{}_N) \in A(K'_1)$ with $|f^1 - g| < \frac{1}{2}\epsilon$ on K'_1. By condition (α), there is a $\delta_1 > 0$ such that if $F \in \mathcal{H}^N$ satisfies $|F - f^1| < \delta_1$ on C_1, then $F \in A(K'_1)$. Since $B^\lambda{}_1 \cup U^\lambda (U^\lambda = \bigcup_i U^\lambda{}_i)$ is X-convex, there are holomorphic functions $'f^2{}_\lambda$, $\lambda = 1, \cdots, N$ with $|'f^2{}_\lambda - f^1{}_\lambda| < \frac{1}{2}\delta_1$ on $C^\lambda{}_1$, $|'f^2{}_\lambda - g_\lambda| < \frac{1}{2}\epsilon_i$ on $K^\lambda{}_i$ for $i^\lambda{}_1 < i \leqq i^\lambda{}_2$. Because $A(K'_2)$ is dense in $\mathcal{H}^N$, there are holomorphic functions $f^2{}_\lambda$, $\lambda = 1, \cdots, N$, on X such that $|f^2{}_\lambda - f^1{}_\lambda| < \frac{1}{2}\delta_1$ on C_1, $|f^2{}_\lambda - g_\lambda| < \frac{1}{2}\epsilon_i$ on $K^\lambda{}_i$, $i_1 < i \leqq i^\lambda{}_2$, with $f^2 = (f^2{}_1, \cdots, f^2{}_N) \in A(K'_2)$.

Let $0 < \delta_2 < \frac{1}{2}\delta_1$ be such that if $F \in \mathcal{H}^N$ satisfies $|F - f^2| < \delta_2$ on C_2, then $F \in A(K'_2)$. Continuing this process, we find a sequence $f^\nu = (f^\nu{}_1, \cdots, f^\nu{}_N) \in A(K'_\nu)$ satisfying the following conditions: $|f^\nu - f^{\nu-1}| < \frac{1}{2}\delta_{\nu-1}$ on $C_{\nu-1}$, $|f_\lambda - g_\lambda| < \frac{1}{2}\epsilon_i$ on $K^\lambda{}_i$, $i^\lambda{}_{\nu-1} < i \leqq i^\lambda{}_\nu$. Let $\delta_\nu < \frac{1}{2}\delta_{\nu-1}$ be such that if $F \in \mathcal{H}^N$ satisfies $|F - f^\nu| < \delta_\nu$ on C_ν, then we have $F \in A(K'_\nu)$. Since $\delta_\nu < \frac{1}{2}\delta_{\nu-1}$, we have $\sum_{\mu=\nu+1}^{\infty} \delta_\mu < \delta_\nu$. Hence if $f = \lim_{\nu\to\infty} f_\nu$, we have $|f - f^\nu| < \delta_\nu$ on C_ν so that $f \in A(K'_\nu)$ for all ν and so $f \in A(K)$ for *any* compact set K of X. Obviously,

if we choose the δ's small enough, we have $|f-g|<\epsilon$ on C, $|f_\lambda - g_\lambda|<\epsilon_i$ on K^λ_i. This completes the proof of the lemma.

4. Imbedding of Stein manifolds. We prove in this section our first main theorem. It is contained in the theorem of the next section but is simpler to prove and depends on sofewhat different ideas.

THEOREM 3. *Let X be a Stein manifold of dimension n. Then the set of one-one regular and proper holomorphic mappings of X into $\boldsymbol{C}^{2n+1}$ is dense in $\mathcal{H}^{2n+1}$.*

LEMMA 3. *Let X be a Stein manifold of dimension n, K a compact set on X. Then, the set of $f\in\mathcal{H}^{2n+1}$ which are one-one and regular on K is dense in $\mathcal{H}^{2n+1}$.*

Proof. Since K is compact, there exists a map $(\phi_1,\cdots,\phi_k): X\to\boldsymbol{C}^k$ (for some k) which is one-one and regular in a neighbourhood of K. Let $(g_1,\cdots,g_{2n+1})$ be arbitrary holomorphic functions on X. The map (g,ϕ): $X\to\boldsymbol{C}^{k+2n+1}$ is then one-one and regular in a neighbourhood of K. It follows from a well known argument of H. Whitney [10] that if we put

$$f_i=\sum_{j=1}^{2n+1}a_{ij}g_i+\sum_{j=2n+2}^{m}a_{ij}\phi_{j-2n-1}$$

$$(m=2n+1+k)\quad i=1,\cdots,2n+1,$$

then $(f_1,\cdots,f_{2n+1})$ is one-one and regular in a neighbourhood of K for a set of matrices (a_{ij}) dense in the space of all $(2n+1)\times m$ matrices. (If V is a submanifold of $\boldsymbol{C}^p$, $p>2n+1$, not necessarily closed, the set of directions so that the projection of $\boldsymbol{C}^p$ along this direction onto $\boldsymbol{C}^{p-1}$ is one-one regular on V is dense in the complex projective space $\boldsymbol{P}^{p-1}$.) The lemma follows easily from this remark.

(Another proof of this lemma can be given on the same lines as the proof of Lemma 6 in Section 6 below.)

LEMMA 4. *If $f\in\mathcal{H}^k$ is one-one, regular in a neighbourhood of a compact set K of the complex manifold X and if $g\in\mathcal{H}^k$ satisfies $|f-g|<\epsilon$ on a neighbourhood K' of K, then if ϵ is small enough, g is one-one and regular on K.*

Proof. Let Δ denote the diagonal of the product space $X\times X$. Clearly, if $\epsilon<\epsilon_0$, then g is regular on K. Moreover, if ϵ_0 is small enough, then there is a neighbourhood of any point of K, depending only on ϵ_0, such that g is

one-one in this neighbourhood. This means that there exists a neighbourhood U of $K \times K \cap \Delta$, depending only on ϵ_0, so that if $|f-g| < \epsilon_0$ on K' and $(x,y) \in U - \Delta$, then $g(x) \neq g(y)$. Let $\delta = \inf_{(x,y) \in K \times K - U} |f(x) - f(y)|$. Then $\delta > 0$; if ϵ_0 is small enough, then clearly $|g(x) - g(y)| \geqq \delta/2 > 0$ if $(x,y) \in K \times K - U$. The result follows.

The proof of Theorem 3 is now easy. Let $A(K) \in \mathfrak{H}^{2n+1}$ be the set of $f \in \mathfrak{H}^{2n+1}$ which are regular and one-one on K. Lemmas 3 and 4 imply that $A(K)$ satisfies all the conditions of Lemma 2. Let now $\{U^\lambda_i\}$, $\lambda = 1, \cdots, 2n+1$, be admissible systems with

$$X = \bigcup_{\lambda=1}^{2n+1} \bigcup_{i=1}^{\infty} U^\lambda_i.$$

Since each U^λ_i is relatively compact, there exist compact sets $K^\lambda_i \subset U^\lambda_i$ with $X = \bigcup_{\lambda,i} K^\lambda_i$. Let $\{B^\lambda_\nu\}$ be an associated sequence of $\{U^\lambda_i\}$, let $g = (g_1, \cdots, g_{2n+1}) \in \mathfrak{H}^{2n+1}$ be given, let C be a compact set of X and let $\epsilon > 0$. We may clearly suppose that $C \subset B^\lambda_1$ for each λ. Then, by Theorem 2, there are holomorphic functions $h_1, \cdots, h_{2n+1}$ on X with

$$|h_\lambda - g_\lambda| < \epsilon/2 \text{ on } C, \quad |h_\lambda| \geqq i+1 \text{ on } K^\lambda_i \text{ for } i \geqq i_0(C).$$

By Lemma 2, there is $f = (f_1, \cdots, f_{2n+1}) \in \mathfrak{H}^{2n+1}$ such that f is one-one, regular on *every* compact set, i.e. on X, satisfying $|f-h| < \epsilon/2$ on C, $|f_\lambda - h_\lambda| < 1$ on K^λ_i for $i \geqq i_0(C)$. The set of points where $|f_\lambda| \leqq i$, $\lambda = 1, \cdots, 2n+1$, is contained in $C \cup \bigcup_{j \leqq \max(i_0(C), i)} \bigcup_{\lambda} K^\lambda_j$ and so is compact. Theorem 3 is proved.

THEOREM 4. *On a Stein manifold X of dimension n, there exist $2n+1$ holomorphic functions f_λ such that if D is X-convex and ϕ holomorphic in D, then ϕ is the limit, uniformly on compact sets of D, of polynomials in the f_λ.*

Proof. Let $f = (f_1, \cdots, f_{2n+1}) : X \to \boldsymbol{C}^{2n+1}$ be a one-one, regular and proper holomorphic map. Since ϕ may be approximated by functions on X, we may suppose that $D = X$. The image M of X under f is a nonsingular, closed submanifold of $\boldsymbol{C}^{2n+1}$. Hence the holomorphic function $\phi \circ f^{-1}$ on M can be extended to a holomorphic function Φ on $\boldsymbol{C}^{2n+1}$ (see [2]). If Φ_k is the sum of the terms of degree $\leqq k$ in the Taylor expansion of Φ about the origin, the Φ_k are polynomials for which $\Phi_k(f_1, \cdots, f_{2n+1})$ converges to ϕ as $k \to \infty$.

5. The case of arbitrary Stein spaces. The next theorem concerns

the imbedding of general Stein spaces; it includes Theorem 3. The proof is however somewhat different.

THEOREM 5. *Let X be a Stein space of dimension n. Then, the set of holomorphic mappings $X \to \boldsymbol{C}^{2n+1}$ which are one-one and proper on X and regular at every uniformizable point of X, is dense in $\mathfrak{H}^{2n+1}$.*

Proof. Let $\phi: X \to \boldsymbol{C}^k$ be any holomorphic map. Let $Y = X \times X$ and let $M'(\phi)$ be the set of $(x, y) \in Y$ with $\phi(x) = \phi(y)$. Clearly, $M'(\phi)$ is an analytic set of Y which contains the diagonal Δ of Y. Let $M(\phi)$ be the union of the irreducible components of $M'(\phi)$ which are not contained in Δ. Then $M(\phi)$ is an analytic set of Y.

Let S be the set of non-uniformizable points of X. Then S is an analytic set in X and $X - S$ is a complex manifold. Let $\rho(\phi, x)$ $(x \in X - S)$ denote the rank of the Jacobian matrix of ϕ with respect to a local coordinate system at x. Let $X'(\phi, m)$ $(m \leqq n)$ denote the set of points of $x \in X - S$ with $\rho(\phi, x) \leqq m$. Clearly $X'(\phi, m)$ is an analytic set in $X - S$. Moreover, by a theorem of Remmert [8, Satz 12], the closure $X(\phi, m)$ of $X'(\phi, m)$ in X is an analytic set in X. Clearly none of its irreducible components is contained in S.

We prove first the following result.

(5.1) Suppose that $\{U_i\}$ is an admissible system, $f_k: X \to \boldsymbol{C}^k (k \geqq 0)$ a holomorphic map satisfying the following two conditions:

$a_k)$ $\dim M(f_k) \leqq 2n - k$

$b_k)$ for $0 \leqq m < n$, we have $\dim X(f_k, m) \leqq n - k + m$.

(Here $\dim B \leqq -1$ means that B is empty.)

Let K_i be a compact subset of U_i and let h be a holomorphic function on X. Let C be a given compact set of X and let $\epsilon > 0$, $\epsilon_i > 0$.

Then, there exists a holomorphic function f on X such that

1) $|h(x) - f(x)| < \epsilon$ for $x \in C$; $< \epsilon_i$ for $x \in K_i$; $i = 1, 2, \cdots$,

2) the map $f_{k+1} = (f_k, f)$ of X into $\boldsymbol{C}^{k+1}$ satisfies $a_{k+1})$ and $b_{k+1})$.

[We remark that conditions $a_0)$ and $b_0)$ are empty.]

To prove (5.1), we proceed as follows: Let M_q, $q = 1, 2, \cdots$, be the irreducible components of $M(f_k)$. Choose on each M_q a point $(x_q, y_q) \notin \Delta$. Then $\{(x_q, y_q)\}$ is a discrete set of $Y = X \times X$.

Let X_m be the union of the irreducible components of $X(f_k, m)$ which

have dimension $= n-k+m$ $(0 \leqq m < n)$. No irreducible component of X_m is contained in S and consequently $\dim(X_m \cap S) < n-k+m$. Also, by hypothesis $b_k)$, $\dim X(f_k, m-1) < n-k+m$. Hence every irreducible component $X^p{}_m$ of X_m $(p=1,2,\cdots)$ contains a point, say $x^p{}_m$, which does not belong to $S \cup X(f_k, m-1)$. Since $x^p{}_m \notin X(f_k, m-1)$, we must have $\rho(f_k, x^p{}_m) = m$. Moreover, the points $\{x^p{}_m\}$, $m=0,1,\cdots,n-1$; $p=1,2,\cdots$, form a discrete subset of X.

We now make two remarks.

1°) *The set of holomorphic functions on X separating finitely many distinct points of X is dense in $\mathfrak{H}$.* For if $x_1,\cdots,x_p$ are the given points, then clearly there is an $f_i \in \mathfrak{H}$ with $f_i(x_j) = \delta_{ij}$; the function $\sum \lambda_i f_i = f$ has the property that $f(x_i) = \lambda_i$; hence this set is not empty. If now $g \in \mathfrak{H}$ is given, then $g+\lambda f$ fails to separate the x_i only if λ is one of the values $(g(x_i)-g(x_j))/(f(x_j)-f(x_i))$ $(i \neq j)$; since λ can be chosen arbitrarily small and different from these finitely many values, the result is proved.

2°) *Let $x_1,\cdots,x_p$ be finitely many points of $X-S$ and let $f: X \to C^k$ be a holomorphic map with $\rho(f, x_i) = \rho_i < n$. Then the set of holomorphic functions $g \in \mathfrak{H}$ such that the map $f' = (f, g): X \to \boldsymbol{C}^{k+1}$ has $\rho(f', x_i) = \rho_i + 1$ for $i = 1,\cdots,p$ is dense in $\mathfrak{H}$.* In the first place, there are holomorphic functions $z_1,\cdots,z_n$ on X forming a coordinate system at each of the points x_i as one sees by taking suitable linear combinations of coordinate systems at the various points consisting of holomorphic functions on X. Again, if $\mu_1,\cdots,\mu_n$ are suitable constants, then the function $\zeta = \mu_1 z_1 + \cdots + \mu_n z_n$ has, together with the given map f, a rank $> \rho$ at x_i. If $g \in \mathfrak{H}$ is arbitrary, then $g+\lambda\zeta$ together with f has rank ρ_i at x_i for some i only if λ is one of finitely many fixed complex numbers. The result follows since λ may be chosen arbitrarily small and $\neq$ these numbers.

Let $x^p{}_m$ and (x_q, y_q) be the points described before 1°) and 2°). We define a subset $A(K)$ of $\mathfrak{H}$ for each compact set $K \subset X$ as follows: $g \in \mathfrak{H}$ belongs to $A(K)$ if the following two conditions are satisfied: (i) if $x^p{}_m \in K$ and f' is the map (f_k, g), then $\rho(f', x^p{}_m) = m+1$; (ii) if $(x_q, y_q) \in K \times K$, then $g(x_q) \neq g(y_q)$. It is obvious, that if $K \subset (K')^0$ and $|g-g_1| < \epsilon$ on K', then $g_1 \in A(K)$. This and 1°) and 2°) above imply that $A(K)$ *satisfies the conditions of Lemma 2 with* $N=1$. Hence, by Lemma 2, if $\{U_i\}$ is the given admissible system, there is a $g \in \mathfrak{H}$ with $|g-h| < \epsilon$ on a given compact set C, $|g-h| < \epsilon_i$ on K_i, $i=1,2,\cdots$ such that $g \in A(K)$ for any compact set K. This last condition clearly means just that $g(x_q) \neq g(y_q)$ for *all* q and that the map $f_{k+1} = (f_k, g)$ has a Jacobian of rank $m+1$ at $x^p{}_m$ for each $m=0,$

$1, \cdots, n-1$ and all p. We assert that the map f_{k+1} satisfies conditions a_{k+1}) and b_{k+1}). In fact, let M be any irreducible components of $M(f_{k+1})$ which is not contained in Δ. Then $M \subset M_q$ for some q (M_q are the irreducible components of $M(f_k)$ not contained in Δ). But $(x_q, y_q) \in M_q$ and clearly $\notin M$; hence $\dim M < \dim M_q \leqq 2n-k$. Hence $\dim M(f_{k+1}) \leqq 2n-(k+1)$ which is a_{k+1}). If A is an irreducible component of $X(f_{k+1}, m)$ then either A is contained in an irreducible component of $X(f_k, m)$ of dimension $< n-k+m$ or A is contained in an irreducible component $X^p{}_m$ of X_m. But then again $x^p{}_m \in X^p{}_m$, $\notin A$ so that $\dim A < n-k+m$. This is b_{k+1}). (5.1) is completely proved.

Theorem 5 is now easily proved. We choose $2n+1$ admissible systems $\{U^\lambda{}_i\}$, $\lambda = 1, \cdots, 2n+1$, and compact sets $K^\lambda{}_i \subset U^\lambda{}_i$ so that $\bigcup_{i,\lambda} K^\lambda{}_i = X$. Let $f = (f_1, \cdots, f_{2n+1}) \in \mathfrak{H}^{2n+1}$ be arbitrary, C a given compact set and $\epsilon > 0$. Then, by Theorem 2, there is a holomorphic function h_λ ($\lambda = 1, \cdots, 2n+1$) with $|h_\lambda - f_\lambda| < \epsilon/2$ on C, $|h_\lambda| \geqq i+1$ on $K^\lambda{}_i$ for $i \geqq i_0(C)$. Because conditions a_0) and b_0) are empty, there is a holomorphic function ϕ_1 on X with $|\phi_1 - h_1| < \epsilon/2$ on C, < 1 on $K^1{}_i$, so that $\dim M(\phi_1) \leqq 2n-1$, $\dim X(\phi_1, m) \leqq n-1+m$ for $0 \leqq m < n$. We the can find successively functions $\phi_2, \cdots, \phi_{2n+1}$ so that if $\Phi_k : X \to \boldsymbol{C}^k$ is the map $(\phi_1, \cdots, \phi_k)$, we have $\dim M(\Phi_k) \leqq 2n-k$, $\dim X(\Phi_k, m) \leqq n-k+m$ for $0 \leqq m < n$ with $|\phi_\lambda - h_\lambda| < \epsilon/2$ on C, < 1 on $K^\lambda{}_i$ for $\lambda \leqq k$. Clearly then the map $\Phi_{2n+1} = \Phi$ is proper and $|\Phi - f| < \epsilon$ on C. Since $\dim M(\Phi) \leqq -1$ and $\dim X(\Phi, m) \leqq -n-1+m \leqq -1$ for $0 \leqq m < n$, Φ is one-one on X and regular on $X - S$. This proves Theorem 5.

From the existence of a proper holomorphic map of a Stein space X into number space, one may deduce the following result:

A closed analytic polyhedron in X (compact set Δ defined as the set of x in a fixed neighbourhood of Δ where $|f_j(x)| \leqq 1$, for finitely many $f_j \in \mathfrak{H}$) is the inverse image *in* X of the closed unit polycylinder in some number space $\boldsymbol{C}^N$ by a holomorphic map $X \to \boldsymbol{C}^N$ (and a corresponding result for open polyhedra).

6. Regular imbedding of Stein spaces. Let X be a complex space and $f : X \to \boldsymbol{C}^k$ a holomorphic map. Let $x_0 \in X$. Suppose that f is one-one in a neighbourhood of x_0. Then, there is a neighbourhood U of x_0 such that $f \mid U$ is a proper map of U into a spherical ball $B \subset \boldsymbol{C}^k$ about $f(x_0)$. According to a well known theorem of Remmert [9], $f(U)$ is an analytic set M in B. As such, it has an induced analytic structure from B.

A map $f: X \to \boldsymbol{C}^k$ is called *regular* at a point $x_0 \in X$ if it is one-one in a neighbourhood of x_0 and if f induces an analytic *isomorphism* of U onto the complex subspace M of B described above. It is said to be regular on a subset of X if it is regular at each point of the subset.

For complex manifolds, the notion of regularity coincides with the statement that the Jacobian matrix of f has rank at x_0 equal to the dimension of X at x_0.

One may define in the same way a regular map of one complex space into another. We then see easily that the composite of regular maps is regular. In particular, if $f: X \to \boldsymbol{C}^k$ is regular at x_0 and F is a map of a neighbourhood of $f(x_0)$ into $\boldsymbol{C}^k$ which is regular at $f(x_0)$, then $F \circ f$ is regular at x_0. Again, if $f: X \to \boldsymbol{C}^k$ is any holomorphic map and there exists a regular map F of a neighbourhood of $f(x_0)$ into $\boldsymbol{C}^k$ so that $F \circ f$ is regular at x_0, then f itself is regular at x_0.

For a complex space X to admit a regular map into some number space $\boldsymbol{C}^k$, it is clear that there must exist an integer $N > 0$ such that for each point $x_0 \in X$, there is a holomorphic map $f: X \to \boldsymbol{C}^N$ which is regular at x_0. If this condition is satisfied, we shall say that X *is of type* N. If n is the dimension of X, then clearly $N \geqq n$; $N = n$ implies that X is a complex manifold.

Not all complex spaces are of type N for some N; there exist even Stein spaces which are not of "finite type." Our next theorem is an analogue of the imbedding theorem for Stein manifolds in the case of those spaces which are locally of bounded type.

THEOREM 6. *Let X be a Stein space of dimension n. Suppose that there exists an integer $N > 0$ such that the space is locally of type N. Then, if $N > n$ the set of one-one, regular, proper holomorphic mappings of X in $\boldsymbol{C}^{N+n}$ is dense in $\mathfrak{M}^{N+n}$.*

Proof. The proof is via two lemmas which are analogous to Lemmas 3 and 4 in the proof of the imbedding of Stein manifolds. For their proof we require the following result which we state before the lemmas. A proof of the theorem is contained in § 8 of the paper [5] by Grauert and Remmert.

Let D be a domain of holomorphy in $\boldsymbol{C}^n$. Let A be an analytic subset of D with the induced analytic structure from D and K a compact set in D. Then, there is a constant $M = M_K$ depending only on K such that if f is a holomorphic function on A with $|f| \leqq 1$ on A, then there is a holomorphic function F in D, $F = f$ on A such that $|F(x)| \leqq M$ for $x \in K$.

LEMMA 5. *Let X be a Stein space and let $K \subset X$ be a compact subset, $K \subset (K')^0$. Let $f \in \mathcal{H}^k$ be one-one and regular on K. Then there is an $\epsilon > 0$ such that if $|f-g| < \epsilon$ on K', where $g \in \mathcal{H}^k$, then g is one-one regular on K.*

Proof. From the proof of Lemma 4, it is clear that we have only to prove the following result:

If f is a one-one proper, regular map of an open set U into a ball $B \subset \boldsymbol{C}^k$, if $x_0 \in U$ and if $|f-g| < \epsilon$, then there is a neighbourhood U' of x_0 depending only on ϵ and not on g (for small ϵ) such that g is one-one and regular on U'.

To prove this, let B' be a smaller ball in B concentric with B. If $z_1, \cdots, z_k$ are the coordinate functions in $\boldsymbol{C}^k$, there is an $\epsilon' > 0$ such that if $H_1, \cdots, H_k$ are holomorphic functions on B with $|H_i - z_i| < \epsilon'$ on B', then $(H_1, \cdots, H_k)$ is one-one and regular in a concentric ball B'' which depends only on ϵ'. If $|f-g| < \epsilon$ on U, then $(f-g) \circ f^{-1}$ is a holomorphic map of $f(U)$ into $\boldsymbol{C}^k$. By the theorem stated above, there is an extension u of $g-f$ to B with $|u| < M\epsilon$ on B'. If $G = z + u$ then clearly we have $|G_i - z_i| < M\epsilon$ on B' $(G = (G_1, \cdots, G_k))$, so that, if $M\epsilon < \epsilon'$, G is one-one regular in B''. By the remark on the composite made at the beginning of the section, $g = G \circ f$ is one-one regular in $U' = f^{-1}(B'')$.

LEMMA 6. *Let X be a Stein space which is locally of type $N > n$. Then the set of $f \in \mathcal{H}^{N+n}$ which are one-one and regular in the neighbourhood of a compact set $K \subset X$ is dense in $\mathcal{H}^{N+n}$.*

Proof. It is clear from Lemma 5 that X itself is of type N. It is sufficient, because of Lemma 5, to prove that any $x_0 \in K$ has a neighbourhood U such that the set of $f \in \mathcal{H}^{N+n}$ which are one-one on K and regular in U is dense in $\mathcal{H}^{N+n}$. Let V be a neighbourhood of x_0 such that there is a map $\phi: X \to \boldsymbol{C}^n$ so that ϕ is a one-one, regular, proper map of V into a ball $B \subset \boldsymbol{C}^N$ about $\phi(x_0)$. Let $A = \phi(U)$. Let $g: X \to \boldsymbol{C}^p$ be an arbitrary holomorphic map. Let $V(g, m)$ $(0 \leqq m < N)$ be defined as follows. Let $g_1 = g \circ \phi^{-1}$ on A. Let $G': B \to \boldsymbol{C}^p$ be a holomorphic map $= g_1$ on A. Then $x \in V$ belongs to $V(g, m)$ if and only if *for any such extension G', the Jacobian of G'* at $\phi(x)$ has rank $\leqq m$. It is clear that $V(g, m)$ is the inverse image in V of the set of points of B where all the G' have rank $\leqq m$. This latter set is defined in B by a family of holomorphic functions *defined everywhere in B*. It is known (see [3, exp. XV and exp. XI]) that such a set is an analytic set. Consequently $V(g, m)$ is an analytic subset of V. Let B' be a smaller concentric ball in B and let $U = \phi^{-1}(B')$. Let $M(g)$ be the set defined in the proof of Theorem 5, viz. the union of the irreducible

components of the set of $(x,y) \in X \times X$ with $g(x) = g(y)$ which are not contained in the diagonal of $X \times X$.

We make the following remark. If $x \in U$ and $g: X \to \boldsymbol{C}^p$ is a holomorphic map so that $g \circ \phi^{-1}$ has an extension G to B of rank $= r < N$ at $\phi(x)$, then the set of $f \in \mathfrak{H}$ so that the map $g' = (g,f)$ has the property that $g' \circ \phi^{-1}$ has an extension G' to B of rank $= r+1$ at $\phi(x)$ is open and dense in $\mathfrak{H}$. To prove this, let H be a holomorphic function on B such that (G,H) has rank $r+1$ at $\phi(x)$. Let h' be a holomorphic function on X with $|h' - H \circ \phi| < \epsilon$ on V. Then $h' \circ \phi^{-1}$ has an extension H' to B with $|H - H'| < M\epsilon$ on B'. Clearly, if ϵ is small enough, (G,H') has rank $r+1$ at $\phi(x)$ so that a function h' exists with the required property. One sees in the same way that the set is open. If $\psi \in \mathfrak{H}$ is arbitrary, Ψ is an extension of $\psi \circ \phi^{-1}$ to B, then if (G,Ψ) has rank r at $\phi(x)$, $(G, \Psi + \lambda H')$ has rank $r+1$ at $\phi(x)$ if $\lambda \neq 0$ which proves our assertion. It now follows, that the following result holds:

Let $g: X \to \boldsymbol{C}^k$ be a holomorphic map satisfying

α_k) the irreducible components of $M(g)$ meeting $K \times K$ have dimension $\leqq 2n - k$.

β_k) The irreducible components of $V(g,m)$ meeting U have dimension $\leqq n - k + m$ for $0 \leqq m < N$.

Then, the set of $h \in \mathfrak{H}$ such that (g,h) satisfies α_{k+1}) and β_{k+1}) is dense in $\mathfrak{H}$.

The proof of this is the same as that of statement (5.1); of course, since U is relatively compact in V, no admissible systems, or Lemma 2, have to be brought in. We omit the details.

Again, since the conditions α_0) and β_0) are empty, it follows that the set of $f \in \mathfrak{H}^{N+n}$ with $\dim(M(f) \cap K \times K) \leqq 2n - n - N (\leqq -1)$ and $\dim(V(f,m) \cap U) \leqq n - (N+n) + m (< 0$ for $0 \leqq m < N)$ is dense in $\mathfrak{H}^{N+n}$. Lemma 6 follows then from the fact that the second condition implies, by the remarks at the beginning of this section, that f is regular.

It may be remarked that the projection method which we used to prove Lemma 3 could also be used to prove Lemma 6.

Theorem 6 follows from Lemmas 2, 5, 6 in the same way that Theorem 3 followed from Lemmas 2, 3, 4.

As in the case of Stein manifolds, Theorem 6 implies that the polynomials in $N+n$ suitable holomorphic functions on a Stein space of dimension n and

type N are dense in $\mathcal{H}$. If on a Stein space X, the polynomials in finitely many functions of are dense in $\mathcal{H}$, then, it is shown easily that X must be of type N for some N.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY.

REFERENCES.

[1] H. Behnke and K. Stein, "Entwicklung analytischer Funktionen auf Riemannschen Flächen," *Mathematische Annalen*, vol. 120 (1948), pp. 430-461.

[2] H. Cartan, "Idéaux et modules des fonctions analytiques de variables complexes," *Bulletin de la Société Mathématique de France*, vol. 78 (1950), pp. 29-64.

[3] ———, *Séminaire E. N. S.*, 1951/52.

[4] H. Grauert, "Charakterisierung der holomorph-vollständigen komplexen Räume," *Mathematische Annalen*, vol. 129 (1955), pp. 233-259.

[5] ——— and R. Remmert, "Komplexe Räume," *ibid.*, vol. 136 (1958), pp. 245-318.

[6] R. Narasimhan, "Imbedding of open Riemann surfaces," *Göttingen Nachrichten*, No. 7 (1960), pp. 159-165.

[7] R. Remmert, "Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes," *Comptes Rendus Hebdomaires des Séances de l'Académie des Sciences*, vol. 243 (1956), pp. 118-121.

[8] ———, "Projektionen analytischer Mengen," *Mathematische Annalen*, vol. 130 (1956), pp. 410-441.

[9] ———, "Holomorphe und meromorphe Abbildungen komplexer Räume," *ibid.*, vol. 133 (1957), pp. 328-370.

[10] H. Whitney, "Differentiable manifolds," *Annals of Mathematics*, vol. 37 (1936), pp. 645-680.

[11] ——— and F. Bruhat, "Quelques propriétés fondamentales des ensembles analytiques réels," *Commentarii Mathematici Helvetici*, vol. 33 (1959), pp. 132-160.

DEFORMATIONS OF COMPACT DIFFERENTIABLE TRANSFORMATION GROUPS.*

By Richard S. Palais and Thomas E. Stewart.†

Let G be a Lie group and M a differentiable ($=C^r, r \geqq 1$) manifold. We recall that a differentiable action of G on M is a differentiable map $\varphi: G \times M \to M$ such that $\varphi(g_1, \varphi(g_2, p)) \equiv \varphi(g_1 g_2, p)$, and if e is the identity of G, then $\varphi(e, p) \equiv p$. By a *deformation* of a differentiable action φ of G on M we mean a one-parameter family φ_t ($t \in I = [0,1]$) of differentiable actions of G on M such that $\varphi_0 = \varphi$ and the map $(g, p, t) \to \varphi_t(g, p)$ of $G \times M \times I$ into M is continuous. If the latter map is differentiable, we say that the deformation φ_t is differentiable. Recall that a deformation of M is a one-parameter family ψ_t ($t \in I$) of diffeomorphisms of M such that ψ_0 is the identity and $(p, t) \to \psi_t(p)$ is a continuous map of $M \times I$ into M. If the latter map is differentiable, then ψ_t is called a differentiable deformation of M. Given a differentiable action φ of G on M and a [differentiable] deformation ψ_t of M we define a [differentiable] deformation φ_t of φ by $\varphi_t(g, p) = \psi_t(\varphi(g, \psi_t^{-1}(p)))$. We call a [differentiable] deformation of φ [*differentiably*] *trivial* if it can be expressed in this form. Recall that two differentiable actions φ_0 and φ_1 of G on M are called equivalent if there exists a diffeomorphism ψ of M such that $\varphi_1(g, p) \equiv \psi(\varphi(g, \psi^{-1}(p)))$. Thus if a deformation φ_t of an action φ is trivial, then φ is equivalent to φ_t for all $t \in I$. Now it is an easy consequence of a theorem proved by one of the authors [1, Theorem 1] that every differentiable action of a Lie group on a Euclidean space with at least one stationary point can be differentiably deformed into a linear action. On the other hand, there is an example of a differentiable (in fact real analytic) action of the circle group on five dimensional Euclidean space which is known not to be equivalent to a linear action. Also, it is a trivial observation that all actions of the line on the torus defined by left translating by a one parameter subgroup are differentiably deformable into each other. However, they clearly are not all equivalent, some being periodic and others not. Thus from the hypothesis that G is compact or that M is

* Received May 23, 1960.

† The authors are National Science Foundation postdoctoral fellows.

compact we cannot conclude that all differentiable deformations of an action of G on M are trivial. However, we will show that if both G and M are compact, then every differentiable deformation of a differentiable action of G on M is in fact differentiably trivial. First, however, we must recall a fairly well-known relation that exists between differentiable deformations of manifolds and differentiable time dependent vector fields. A differentiable time dependent vector field on M is a family $X(t)$ $(t \in I)$ of vector fields on M such that $(p, t) \to X(t)_p$ is a differentiable map of $M \times I$ into the tangent bundle of M (equivalently, in local coordinates the components of $X(t)_p$ are jointly differentiable in t and the coordinates of p). A differentiable path $\gamma: I \to M$ is called an integral curve of $X(t)$ if its tangent at t is $X(t)_{\gamma(t)}$ for all $t \in I$. This is equivalent to the requirement that $t \to (\gamma(t), t)$ be an integral curve of the vector field X^* on $M \times I$ given by $X^*_{(p,t)} = (X(t)_p, D_t)$. If ψ_t is a deformation of M, then we define a differentiable time dependent vector field $X(t)$ on M by $X(t)_{\psi_t(p)} =$ tangent to $s \to \psi_s(p)$ at $s = t$. The uniqueness theorem for ordinary differential equations implies that ψ_t can be recovered from X, for $t \to \psi_t(p)$ is the integral curve of X starting at p. If M is not compact, then not every differentiable time dependent vector field $X(t)$ arises in this way; in fact, there will not in general be an integral curve of $X(t)$ starting at an arbitrary point p of M and defined for the whole unit interval. However, if M is compact, this pathology cannot arise and every differentiable time dependent vector field on M generates a differentiable deformation of M. We now prove

THEOREM. *If φ is a differentiable action of a compact Lie group G on the compact differentiable manifold M, then any differentiable deformation of φ is differentiably trivial.*

Proof. If φ_t is a differentiable deformation of φ, then $\Phi: G \times M \times I \to M \times I$ defined by $\Phi(g, (p, t)) = (\varphi_t(g, p), t)$ is clearly a differentiable action of G on $M \times I$. If we write Φ_g for the diffeomorphism $(p, t) \to \Phi(g, (p, t))$ of $M \times I$ and π for the projection of $M \times I$ on I, then $\pi\Phi_g = \pi$ for all $g \in G$. Let Δ denote the vector field $(0, D)$ on $M \times I$. Since G is compact, we can "average" Δ over G, i.e. form the vector field Δ^* defined by $\Delta^*_{(p,t)} = \int \delta\Phi_g(\Delta_{\Phi_g{}^{-1}(p,t)})\, d\mu(g)$ where μ is Haar measure on G. Since Φ is differentiable, Δ^* is a differentiable vector field. By invariance of Haar measure Δ^* is invariant, in the sense that $\delta\Phi_g(\Delta^*_{(p,t)}) = \Delta^*_{\Phi_g(p,t)}$. Because $\delta\pi\delta\Phi_g = \delta\pi$ for all $g \in G$ and $\delta\pi(\Delta_{\Phi_g{}^{-1}(p,t)}) = D_t$, it follows that $\delta\pi(\Delta^*_{(p,t)}) = D_t$, and therefore $\Delta^*_{(p,t)} = (X(t)_p, D_t)$ for some differentiable time dependent vector field $X(t)$ on M. If ψ_t is the corresponding differentiable deformation of M,

then $(\psi_t(p), t)$ is the integral curve of Δ^* starting at $(p, 0)$. Since Δ^* is invariant, it follows that $\Phi_g(\psi_t(p), t)$ is the integral curve of Δ^* starting at $\Phi_g(p, 0) = (\varphi_0(g, p), 0) = (\varphi(g, p), 0)$. But $\Phi_g(\psi_t(p), t) = (\varphi_t(g, \psi_t(p)), t)$ and the integral curve of Δ^* starting at $(\varphi(g, p), 0)$ is $(\psi_t(\varphi(g, p)), t)$, so $\varphi_t(g, \psi_t(p)) = \psi_t(\varphi(g, p))$. Replacing p by $\psi_t^{-1}(p)$ we get $\varphi_t(g, p) = \psi_t(\varphi(g, \psi_t^{-1}(p)))$. q. e. d.

The notion of a deformation of a general group action on a space can be defined, of course, exactly as above replacing all hypothesis of differentiability by continuity, diffeomorphism by homeomorphism, etc. However, the above theorem is false then even under the most stringent demands short of the assumptions of the theorem (indeed "almost all" actions of compact groups on a sphere can be deformed to linear actions). In a sense this is fortunate since we know that the conclusion of the theorem, i. e. two actions that can be deformed into one another are equivalent by conjugation, is too demanding. Saying that two actions φ, φ' are D-equivalent if there is a deformation φ_t such that $\varphi_0 = \varphi$, $\varphi_1 = \varphi'$ provides us with a partition of all action of a compact group G on M which is conceivably more convenient for many of the theorems of transformation groups. Assuming that the group G is compact and the space M on which it acts is a compact, generalized manifold the following problems seem reasonable:

(1) If φ and φ' are D-equivalent actions of G on M, are the orbit spaces of the same homotopy type?

(2) If φ and φ' are D-equivalent, what is the relation of their orbit structures? In particular, are the fixed point sets of the same homotopy type?

It is not even clear in (2) that if φ has fixed points, then the same must be true of φ'.

THE INSTITUTE FOR ADVANCED STUDY.

REFERENCES.

[1] T. E. Stewart, "On groups of diffeomorphisms," *Proceedings of the American Mathematical Society*, vol. 11 (1960), pp. 559-563.

INDEX

PAGE

ABHYANKAR, SHREERAM. Tame coverings and fundamental groups of algebraic varieties, II, ... 120

ABHYANKAR, SHREERAM. Tame coverings and fundamental groups of algebraic varieties, III, ... 179

ABHYANKAR, SHREERAM. Tame coverings and fundamental groups of algebraic varieties. Part IV: Product theorems, . . 341

ABHYANKAR, SHREERAM. Tame coverings and fundamental groups of algebraic varieties. Part V: Three cuspidal plane quartics, 365

ABHYANKAR, SHREERAM. Tame coverings and fundamental groups of of algebraic varieties. Part VI: Plane curves of order at most four, ... 374

AUSLANDER, LOUIS. Solvable Lie groups acting on manifolds, . . 653

AUSLANDER, LOUIS. Fundamental groups of compact solvmanifolds, . 689

BADÉ, WILLIAM G. and PHILIP C. CURTIS, JR. Homomorphisms of commutative Banach algebras, ... 589

BADÉ, WILLIAM G. and PHILIP C. CURTIS, JR. The Wedderburn decomposition of commutative Banach algebras, . . . 851

BOREL, A. and F. HIRZEBRUCH. Characteristic classes and homogeneous spaces, III, ... 491

CHACON, RAFAEL V. Limit properties at zero on the Markov group, 106

CHERN, SHIING-SHEN. Complex analytic mappings of Riemann surfaces, ... 323

CHEVALLEY, C. Sur la théorie de la variété de Picard, . . . 435

CLIFFORD, A. H. Basic representations of completely simple semigroups, ... 430

COHEN, PAUL J. On a conjecture of Littlewood and idempotent measures, ... 191

COHEN, PAUL J. On homomorphisms of group algebras, . . 213

COHN, HARVEY. Decomposition into four integral squares in the fields of $2^{\frac{1}{3}}$ and $3^{\frac{1}{3}}$, ... 301

CORDES, H. O. On maximal first order partial differential operators, 63

PAGE

CURTIS, PHILIP C., JR. See William G. Badé and Philip C. Curtis, Jr., page 589.

CURTIS, PHILIP C., JR. See William G. Badé and Philip C. Curtis, Jr., page 851.

DWORK, BERNARD. On the rationality of the zeta function of an algebraic variety, ... 631

EHRENPREIS, L. Solution of some problems of division. Part IV. Invertible and elliptic operators, ... 522

FEIT, WALTER. A characterization of the simple groups $SL(2, 2^a)$, . 281

FOGUEL, S. R. Finite dimensional perturbations in Banach spaces, . 260

GREEN, JOHN W. Functions that are harmonic or zero, . . . 867

GREENBERG, LEON. Conformal transformations of Riemann surfaces, . 749

HAMSTROM, MARY-ELIZABETH. Regular mappings whose inverses are 3-cells, ... 393

HIRONAKA, HEISUKE. A generalized theorem of Krull-Seidenberg on parametrized algebras of finite type, ... 831

HIRZEBRUCH, F. See A. Borel and F. Hirzebruch, page 491.

HUGHES, D. R. Collineation groups of non-Desarguesian planes, II, 113

HUGHES, D. R. and ERWIN KLEINFELD. Seminuclear extensions of Galois fields, ... 389

KADISON, RICHARD V. and I. M. SINGER. Triangular operator algebras. Fundamentals and hyperreducible theory, ... 227

KLEINFELD, ERWIN. See D. R. Hughes and Erwin Kleinfeld, page 389.

KOLCHIN, E. R. Abelian extensions of differential fields, . . . 779

KUNZE, R. A. and E. M. STEIN. Uniformly bounded representations and harmonic analysis of the 2×2 real unimodular group, 1

MASSEY, W. S. On the Stiefel-Whitney classes of a manifold, . . 92

MEYER, JEAN-PIERRE. Whitehead products and Postnikov systems, . 271

MILNOR, J. On the cobordism ring Ω^* and a complex analogue, Part I, 505

NARASIMHAN, RAGHAVAN. Imbedding of holomorphically complete complex spaces, ... 917

NEUWIRTH, LEE. The algebraic determination of the genus of knots, 791

NEWMAN, MORRIS. Subgroups of the modular group and sums of squares, ... 761

OWEN, O. G. An ultrahyperbolic equation with an integral condition, 799

PAGE

PALAIS, RICHARD S. and THOMAS E. STEWART. Deformations of compact differentiable transformation groups, ... 935

PETERSON, FRANKLIN P. Whitehead products and the cohomology structure of principal fibre spaces, ... 649

RADEMACHER, HANS. A proof of a theorem on modular functions, . 338

REMMERT, REINHOLD. Analytic and algebraic dependence of meromorphic functions, ... 891

SACKSTEDER, RICHARD. On hypersurfaces with no negative sectional curvatures, ... 609

SAH, CHIH-HAN. Quadratic forms over fields of characteristic 2, . 812

SINGER, I. M. See Richard V. Kadison and I. M. Singer, page 227.

SMITH, J. WOLFGANG. Fundamental groups on a Lorentz manifold, 873

STEIN, E. M. See R. A. Kunze and E. M. Stein, page 1.

STEWART, THOMAS E. See Richard S. Palais and Thomas E. Stewart, page 935.

STONE, A. H. Hereditarily compact spaces, ... 900

WANG, HSIEN-CHUNG. Compact transformation groups of S^n wtih an $(n-1)$-dimensional orbit, ... 698

WERMER, JOHN. On a subalgebra of $L(-\infty, \infty)$, ... 103

WOLF, JOSEPH A. The manifolds covered by Riemannian homogeneous manifold, ... 661

ON THE PREPARATION OF MANUSCRIPTS.

The following instructions are suggested or dictated by the necessities of the technical production of the American Journal of Mathematics. Authors are urged to comply with these instructions, which have been prepared in their interests.

Manuscripts not complying with the standards usually have to be returned to the authors for typographic explanation or revisions and the resulting delay often necessitates the deferment of the publication of the paper to a later issue of the Journal.

Horizontal fraction signs should be avoided. Instead of them use either solidus signs / or negative exponents.

Neither a solidus nor a negative exponent is needed in the symbols $\frac{1}{2}$, $\frac{1}{\pi}$, $\frac{1}{2\pi}$, $\frac{1}{2\pi i}$, which are available in regular size type.

Binomial coefficients should be denoted by C_k^n and not by parentheses. Correspondingly, for symbols of the type of a quadratic residue character the use of some non-vertical arrangement is usually imperative.

For square roots use either the exponent $\frac{1}{2}$ or the sign $\surd$ without the top line, as in $\surd -1$ or $\surd(a+b)$.

Replace $e^{(\ )}$ by $\exp(\ )$ if the expression in the parenthesis is complicated.

By an appropriate choice of notations, avoid unnecessary displays.

Simple formulae, such as $A+iB=\frac{1}{2}C^*$ or $s_n=a_1+\cdots+a_n$, should not be displayed (unless they need a formula number).

Use $'$ or $d\ /dx$, possibly D, but preferably not a dot, in order to denote ordinary differentiation and, as far as possible, a subscript in order to denote partial differentiation (when the symbol ∂ cannot be avoided, it should be used as $\partial\ /\partial x$).

Commas between indices are usually superfluous and should be avoided if possible.

In a determinant use a notation which reduces it to the form $\det a_{ik}$.

Subscripts and superscripts cannot be printed in the same vertical column, hence the manuscript should be clear on whether a_i^k or $a^k{}_i$ is preferred. (Correspondingly, the limits of summation must not be typed *after* the Σ-sign, unless either Σ_1^m or $\Sigma^m{}_1$ is desired.) If a letter carrying a subscript has a prime, indicate whether x_i' or x'_i is desired.

Experience shows that a tilde or anything else *over* a letter is very unsatisfactory. Such symbols often drop out of the type after proof-reading and, when they do not, they usually appear uneven in print. For these reasons we advise against their use. This advice applies also to a bar over a Greek or German letter (for the symbol of complex conjugation an asterisk is often allowed by the context). Type carrying bars over ordinary size italic letters of the Latin alphabet is available.

Bars reaching over several letters should in any case be avoided (in particular, type lim sup and lim inf instead of lim with upper and lower bars).

Repeated subscripts and superscripts should be used only when they cannot be avoided. since the index of the principal index usually appears about as large as the principal index. Bars and other devices *over* indices cannot be supplied. On the other hand, an asterisk or a prime (to be printed *after* the subscript) is possible on a subscript. The same holds true for superscripts.

Distinguish carefully between l. c. "oh," cap. "oh" and zero. One way of distinguishing them is by underlining one or two of them in different colors and explaining the meaning of the colors.

Distinguish between ϵ (epsilon) and $\in$ or ε (symbol), between x (eks) and $\times$ (multiplication sign), between l. c. and cap. phi, between l. c. and cap. psi, between l. c. k and kappa and between "ell" and "one" (for the latter, use $\mathfrak{k}$ and $\mathfrak{l}$ respectively).

Avoid unnecessary footnotes. For instance, references can be incorporated into the text (parenthetically, when necessary) by quoting the number in the bibliographic list, which appear at the end of the paper. Thus: "[3], pp. 261-266."

Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Societé Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

In any case, the references should be precise, unambiguous and intelligible.

Usually sections numbers and section titles are printed in bold face, the titles "Theorem," "Lemma" and "Corollary" are in caps and small caps, "Proof," "Remark" and "Definition" are in italics. This (or a corresponding preference) should be marked in the manuscript. Use a period, and *not* a colon, after the titles Theorem, Lemma, etc.

German, script and bold face letters should be underlined in various colors and the meaning of the colors should be explained. The same device is needed for Greek letters if there is a chance of ambiguity. In general, mark all cap. Greek letters.

All instructions and explanations for the printer can conveniently be collected on a separate sheet, to be attached to the manuscript.

In case of doubt, recent issues of the Journal may be consulted.

Prices Effective January 1960

American Journal of Hygiene. Edited by ABRAHAM G. OSLER, Managing Editor, A. M. BAETJER, MANFRED M. MAYER, R. M. HERRIOTT, F. B. BANG, P. E. SARTWELL, and ERNEST L. STEBBINS. Publishing two volumes of three numbers each year, volume 69 is now in progress. Subscription $12 per year. (Foreign postage, 50 cents; Canadian postage, 25 cents.)

American Journal of Mathematics. Edited by W. L. CHOW, J. A. DIEUDONNÉ, A. M. GLEASON and PHILIP HARTMAN. Quarterly. Volume 81 in progress. $11.00 per year. (Foreign postage, 60 cents; Canadian, 30 cents.)

American Journal of Philology. Edited by H. T. ROWELL, LUDWIG EDELSTEIN, JAMES W. POULTNEY, JOHN H. YOUNG, JAMES H. OLIVER, and EVELYN H. CLIFT, Secretary. Volume 85 is in progress. $6.00 per year. (Foreign postage, 50 cents; Canadian, 25 cents.)

Bulletin of the History of Medicine. OWSEI TEMKIN, Editor. Bi-monthly. Volume 33 in progress. Subscription $6 per year. (Foreign postage, 50 cents; Canadian 25 cents.)

Bulletin of the Johns Hopkins Hospital. PHILIP F. WAGLEY, Managing Editor. Monthly. Subscription $10.00 per year. Volume 104 is in progress. (Foreign postage, 50 cents; Canadian, 25 cents.)

ELH. A Journal of English Literary History. Edited by D. C. ALLEN (Senior Editor), G. E. BENTLEY, JACKSON I. COPE, RICHARD HAMILTON GREEN, J. HILLIS MILLER, ROY H. PEARCE, and E. R. WASSERMAN. Quarterly. Volume 25 in progress. $6.00 per year. (Foreign postage, 40 cents; Canadian, 20 cents.)

Johns Hopkins Studies in Romance Literatures and Languages. Seventy-six numbers have been published.

Johns Hopkins University Studies in Archaeology. Thirty-nine volumes have appeared.

Johns Hopkins University Studies in Geology. Seventeen numbers have been published.

Johns Hopkins University Studies in Historical and Political Science. Under the direction of the Departments of History, Political Economy and Political Science. Volume 77 in progress. $6.50.

Modern Language Notes. NATHAN EDELMAN, General Editor. Eight times yearly. Volume 74 in progress. $8.00 per year. (Foreign postage, 60 cents; Canadian, 30 cents.)

A complete list of publications will be sent on request

There was published in 1956 a 64-page

INDEX

to volumes 51-75 (1929-1953) of

THE AMERICAN JOURNAL OF MATHEMATICS

The price is $2.50. A 60-page Index to volumes 1-50 (1879-1928), published in 1932, is also available. The price is $3.00.

Copies may be ordered from The Johns Hopkins Press, Baltimore 18, Maryland.